

Advanced Model Checking
Summer term 2014

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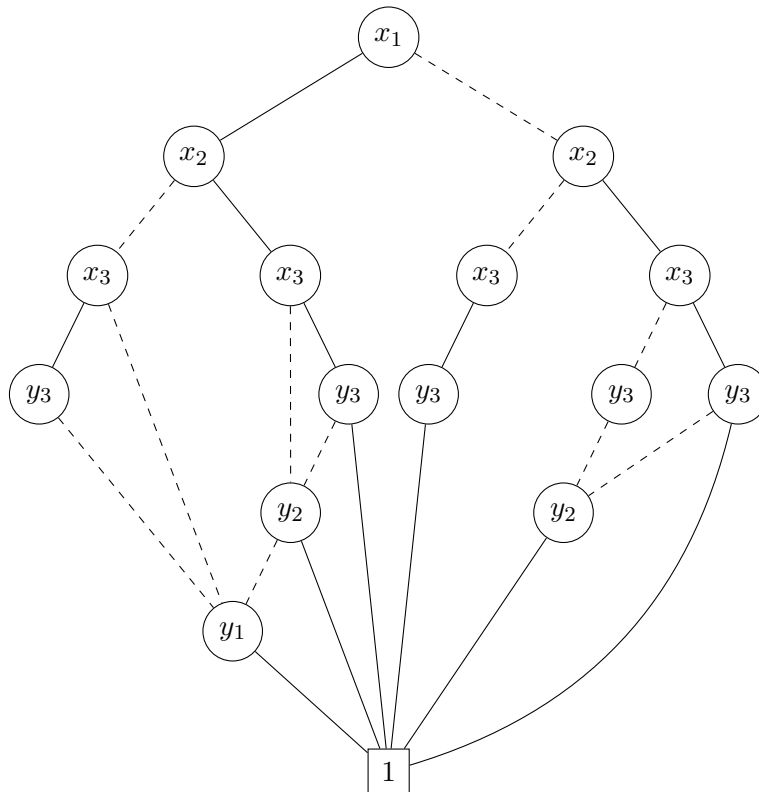
– Series 9 –

Hand in on **25 June** before the exercise class.

Exercise 1

(2 points)

Given the following ROBDD, determine the boolean function $f(x_1, x_2, x_3, y_1, y_2, y_3)$ that it represents.



Hint: it might be worthwhile to find a better variable ordering first.

Solution: The boolean function is

$$f(x_1, x_2, x_3, y_1, y_2, y_3) = \left(\bigvee_{i=1}^3 (x_i \wedge y_i) \right) \wedge \neg(x_1 \wedge \neg x_2 \wedge x_3 \wedge y_3) \wedge \neg(\neg x_1 \wedge x_2 \wedge \neg x_3 \wedge y_3 \wedge y_2).$$

Exercise 2

(3 points)

Consider the family of functions (for $k > 0$) given by

$$f_k(x_0, \dots, x_{n-1}, a_0, \dots, a_{k-1}) = x_{|a|}$$

where $n = 2^k$, $x_i \in \{0, 1\}$ for $0 \leq i < n$ and $a_j \in \{0, 1\}$ for $0 \leq j < k$ and $|a| = \sum_{j=0}^{k-1} a_j 2^j$. In other words, the function returns the $|a|$ -th bit of $(x_1 \dots x_n)$.

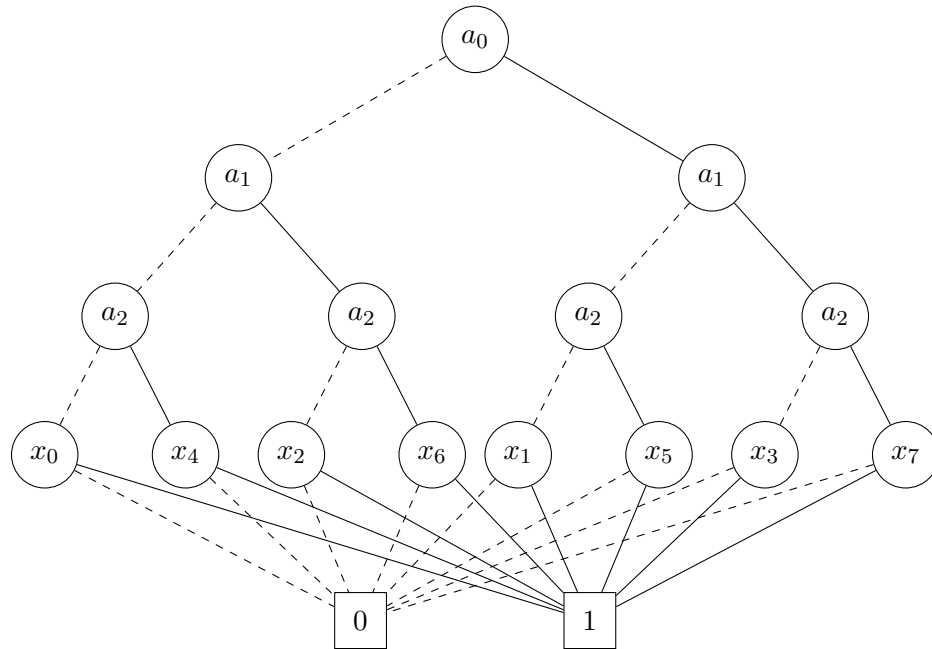
Provide the ROBDDs that represent f_3 for the following two variable orderings:

(i) $a_0 < \dots < a_{k-1} < x_0 < \dots < x_{n-1}$,

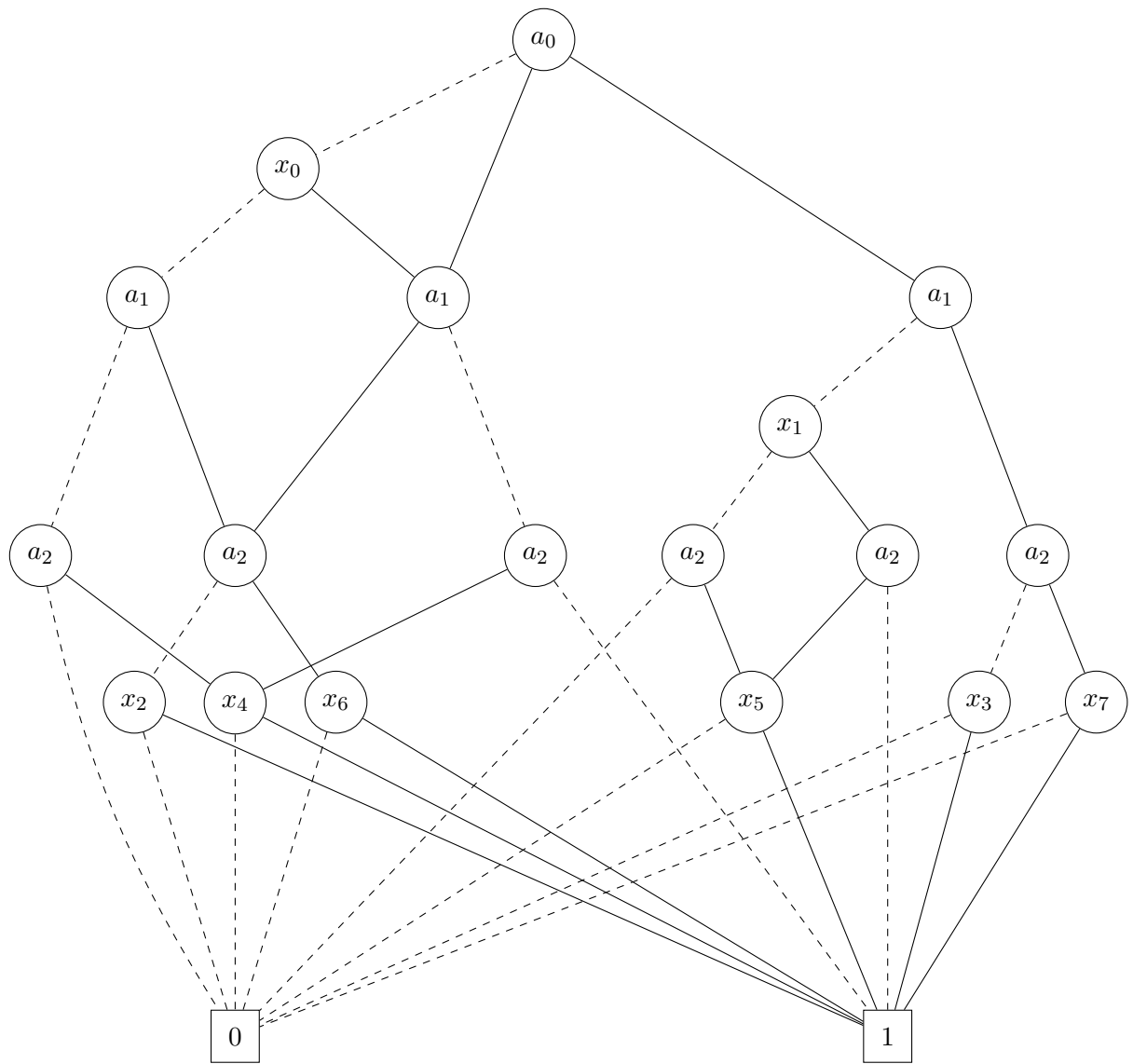
(ii) $a_0 < x_0 < a_1 < x_1 \dots < a_{k-1} < x_{k-1} < x_k < \dots < x_{n-1}$.

Solution:

(i) The ROBDD for this variable ordering is depicted below.



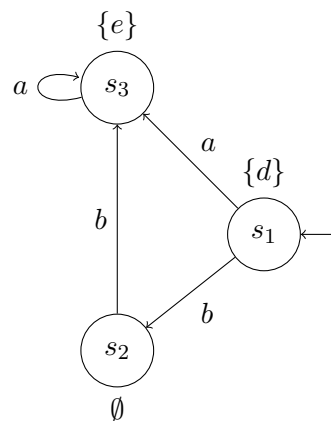
(ii) The ROBDD for this variable ordering is depicted below.



Exercise 3

(2 points)

Consider the following transition system.



- (i) Define the switching functions that represent the transition system (as defined in the lecture/book).
- (ii) Encode the switching function for the transition relation using an ROBDD.

Solution:

- (i) Let $S = Eval(\bar{x})$, where $\bar{x} = (x_1, x_2)$ consists of two Boolean variables. We identify s_1 with $(0, 0) \in Eval(\bar{x})$, s_2 with $(1, 0) \in Eval(\bar{x})$ and s_3 with $(0, 1) \in Eval(\bar{x})$. Then the states can be encoded by

- $\chi_{s_1}(s) = \neg x_1 \wedge \neg x_2$,
- $\chi_{s_2}(s) = x_1 \wedge \neg x_2$,
- $\chi_{s_3}(s) = \neg x_1 \wedge x_2$.

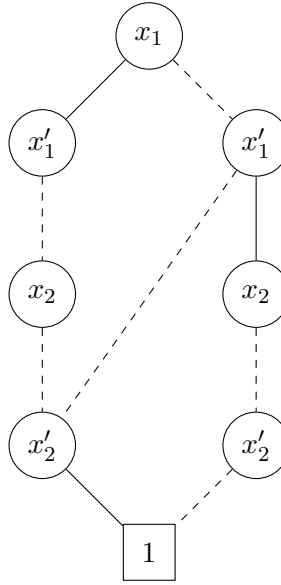
The functions $\chi_{Sat(d)}$ and $\chi_{Sat(e)}$ can be given as

- $\chi_{Sat(d)}(s) = \chi_{s_1}(s)$,
- $\chi_{Sat(e)}(s) = \chi_{s_3}(s)$.

Finally, the transition relation $\Delta: Eval(\bar{x}, \bar{x}') \rightarrow \{0, 1\}$ can be given as

$$\Delta(s, s') = (\chi_{s_1}(s) \wedge \chi_{s_2}(s')) \vee (\chi_{s_1}(s) \wedge \chi_{s_3}(s')) \vee (\chi_{s_2}(s) \wedge \chi_{s_3}(s')) \vee (\chi_{s_3}(s) \wedge \chi_{s_3}(s'))$$

- (ii) We choose the variable ordering $x_1 < x'_1 < x_2 < x'_2$ and obtain the following ROBDD for the transition relation.



Exercise 4

(3 points)

Let $\bar{x}$, $\bar{x}'$ and $\bar{b}$ be three vectors of Boolean variables of size $n > 0$ and let a_i denote the i -th variable in a vector $\bar{a}$. Assume that the transition relation of a transition system TS with state space $S = Eval(\bar{x})$ is given by means of a switching function $\Delta: Eval(\bar{x}, \bar{x}') \rightarrow \{0, 1\}$ (as seen in the lecture). Let $Q = \{B_0, \dots, B_{2^n-1}\}$ be a partition of the state space ($S = \bigcup_{i=0}^{2^n-1} B_i$ and $B_i = \emptyset$ is possible) represented as a switching function $f_Q: Eval(\bar{x}, \bar{b}) \rightarrow \{0, 1\}$ defined by

$$f_Q(s, B_i) = 1 \iff s \in B_i.$$

- (i) Using the available switching functions and the usual operations on them, define another switching function $f_{sigQ}: Eval(\bar{x}, \bar{b}) \rightarrow \{0, 1\}$ given by

$$f_{sigQ}(s, B) = 1 \iff \exists s' \in Post(s) . s' \in B.$$

- (ii) Let the characteristic functions $\chi_{Sat(a)}$ and $\chi_{Sat(b)}$ for the only two atomic propositions $a, b \in AP$ in TS be given. Explain how the coarsest partition that respects the labeling can be computed in terms of a switching function. In other words, how can a function $f_{Q^*}: Eval(\bar{x}, \bar{b}) \rightarrow \{0, 1\}$ that represents the partition $Q^* = \{\{s \mid L(s) = A\} \mid A \in 2^{AP}\}$ in the way described earlier, be derived? You may use the usual operations on switching functions as well as the “special” switching functions $i|_{\bar{b}}: Eval(\bar{b}) \rightarrow \{0, 1\}$ that evaluate to 1 if and only if the input is the binary encoding of the number i .

- (iii) Conceptually explain how the functions $f_Q(s, B_i)$ and $f_{\text{sig}^Q}(s, B_i)$ can be used to compute the coarsest bisimulation equivalence on TS .
- (iv) Assuming that all switching functions are stored as ROBDDs using the variable ordering $x_1 < x'_1 < x_2 < \dots < x_n < x'_n < b_1 < \dots < b_n$, how can the proposed method of (iii) be implemented cleverly?
Hint: How can the reducedness of the ROBDD(s) be exploited?

Solution:

- (i) $f_{\text{sig}^Q}(s, B_i) = \exists s' (\Delta(s, s') \wedge (f_Q(s, B_i)\{s' \leftarrow s\}))$
- (ii) $f_{Q^*}(s, B) = (\neg \chi_{\text{Sat}(a)} \wedge \neg \chi_{\text{Sat}(a)} \wedge 0|_{\bar{b}}) \vee (\neg \chi_{\text{Sat}(a)} \wedge \chi_{\text{Sat}(a)} \wedge 1|_{\bar{b}}) \vee (\chi_{\text{Sat}(a)} \wedge \neg \chi_{\text{Sat}(a)} \wedge 2|_{\bar{b}}) \vee (\chi_{\text{Sat}(a)} \wedge \chi_{\text{Sat}(a)} \wedge 3|_{\bar{b}})$
- (iii) • First, observe that the function $f_{Q^*}(s, B)$ encodes the initial partition of the usual partition refinement approach for computing the bisimulation equivalence. Secondly, the switching function $f_{\text{sig}^Q}(s, B_i)$ encodes the *signature* $\text{sig}^Q(s) = \{B \in Q \mid \exists s' \in \text{Post}(s). s' \in B\}$ of a state s .
- Now observe that two states s and s' with $\text{sig}^{Q^*}(s) \neq \text{sig}^{Q^*}(s')$ cannot be bisimilar. Thus, a new partition Q' can be built by grouping all states with equal signature. Put differently, iteratively computing

$$Q^0 := Q^* \quad Q^{n+1} := \{\{t \mid \text{sig}^{Q^n}(t) = \text{sig}^{Q^n}(s)\} \mid s \in S\}$$

until it stabilizes ($Q^m = Q^{m+1} = \dots$) produces the partition Q^m that is the coarsest bisimulation equivalence on TS . Since we can already compute switching functions for the signature and the partition ROBDDs, it remains to discuss how we can group states with equal signature to obtain the ROBDD for $f_{Q^{n+1}}$ given an ROBDD for f_{Q^n} .

- (iv) • For this, observe that there are no two nodes in the ROBDD for $f_{\text{sig}^{Q^n}}$ that represent exactly the same function, because it is *reduced*. Since the variables encoding s are all above the variables of B , this means that two states have the same signature if and only if following their encodings in the ROBDD leads to *exactly the same node*.
- Consequently, a depth-first traversal of the ROBDD that replaces the topmost nodes that belong to the variables in $\bar{b}$ with a new ROBDD that encodes a not-yet-used index in the new partition, yields a new ROBDD that represents $f_{\text{sig}^{Q^{n+1}}}$ (illustration in the exercise class).