

Series 1 - Solution

① To proof: T_i memoryless, i.e.

$$P\{T = k+m \mid T > m\} = P\{T = k\}, \quad k \geq 1$$

$$P\{T = k+m \mid T > m\}$$

cond. prob.: $P(A|B) = \frac{P(AB)}{P(B)}$

$$= \frac{P\{T = k+m \mid T > m\}}{P\{T > m\}}$$

redundant, $k \geq 1$

$$= \frac{P\{T = k+m\}}{P\{T > m\}}$$

Use prob. distribution of T :

$$P\{T_i = k\} = (1-p)^{k-1} \cdot p, \quad k \geq 1$$

$$= \frac{p \cdot (1-p)^{k+m-1}}{\sum_{i=m+1}^{\infty} (1-p)^{i-1} \cdot p}$$

$$(1-p)^m + (1-p)^{m+1} + \dots$$

$$= (1-p)^m \cdot (1 + (1-p) + (1-p)^2 + \dots)$$

$$= \frac{(1-p)^{k+m-1}}{(1-p)^m \cdot \sum_{i=0}^{\infty} (1-p)^i}$$

geometric series

$$= \frac{1}{1-(1-p)}$$

$$= \frac{(1-p)^{k-1}}{1}$$

$$= p \cdot (1-p)^{k-1} = P\{T = k\}$$

geometric distribution



(2)

a)

- state s is transient if there is a positive probability that s is never visited again.

Let $p_{s,s'}(n)$ be the n -step transition probability for move from s to s' in n steps.

Intuition: $s \rightsquigarrow s'$
 ~~$s \rightsquigarrow s$~~ $\rightsquigarrow \dots$

Def: s transient iff

$$\exists s' \neq s, \exists n > 0 \left(p_{s,s'}(n) > 0 \wedge \forall m > 0, p_{s,s}(m) = 0 \right)$$

Alternative: Define suitable random variables.

- S recurrent iff s is not transient!
- all states that are not transient are ~~recurrent~~ recurrent.

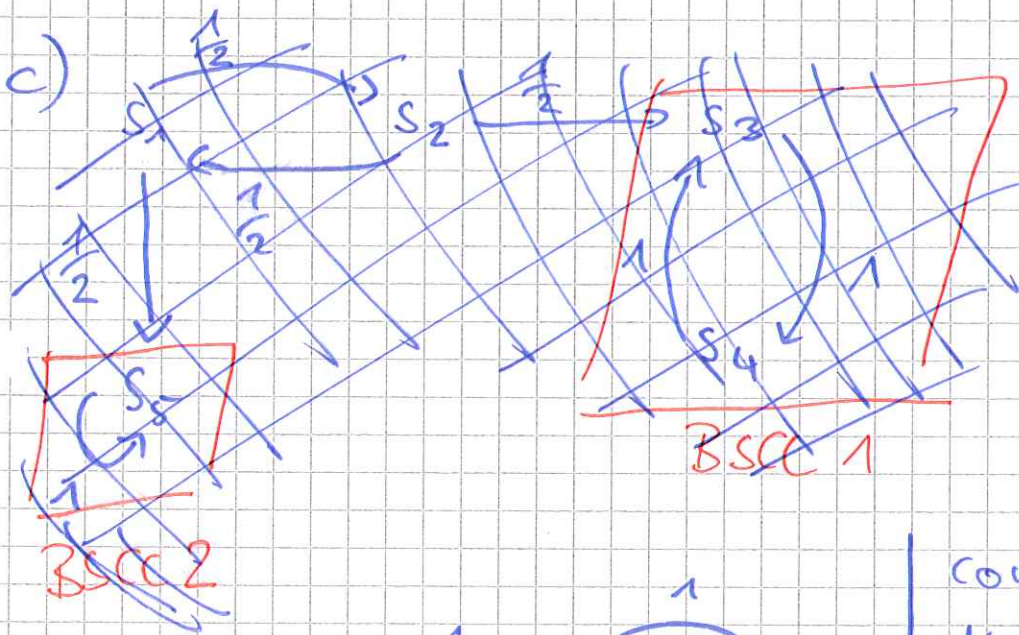
b) Determine the recurrent states of DTMC D .
(finite state space!)

1. Use Tarjan's algorithm to determine the BSCCs of the induced graph of D .

$$O(m+n) !$$

↑ number of transitions
↑ number of states
(positive probabilities)

2. All states in BSCCs are recurrent, all other ones are transient.



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I - P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Computing limiting distribution:

$$\underline{v} = \underline{v} \cdot P$$

$$\underline{v} \cdot (I - P) = \underline{0}, \sum_i v_i = 1$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \underline{0}$$

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$$x_1 = 0$$

$$-x_1 + x_2 - x_3 = 0 \Rightarrow x_2 = x_3$$

$$-x_2 + x_3 = 0$$

and $x_2 + x_3 = 1$

$$\Rightarrow x_2 = x_3 = \frac{1}{2}$$

Alternative

$$\underline{v} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \underline{v}$$

$$\Rightarrow v_1 = 0$$

$$v_2 = v_3 (+v_1)$$

$$v_3 = v_2$$

Exercise 3

(3 points)

- a) The one-step transition probability matrix P is as follows:

$$P = \begin{pmatrix} 0 & 1/4 & 0 & 3/4 \\ 1/2 & 0 & 1/3 & 1/6 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 1/4 & 1/4 \end{pmatrix}$$

P^3 is needed!

And the two-step and three-step transition probability matrix P^2 and P^3 are respectively:

$$P^2 = \begin{pmatrix} 1/8 & 3/8 & 13/48 & 11/48 \\ 0 & 5/24 & 3/8 & 5/12 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/8 & 23/48 & 7/48 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 3/16 & 7/48 & 26/64 & 41/192 \\ 5/48 & 5/24 & 79/144 & 5/36 \\ 0 & 0 & 1 & 0 \\ 1/16 & 13/96 & 107/192 & 47/192 \end{pmatrix}$$

The entry $P_{0,3}^3 = 41/192$ is the probability of going from s_0 to s_3 in exactly 3 steps. An alternative way to obtain this probability is to consider all the paths that go from s_0 to s_3 in exactly 3 steps and add up their probabilities. There are four such paths:

$$\Pr\{s_0 s_1 s_0 s_3\} = 3/32 \quad \Pr\{s_0 s_1 s_3 s_3\} = 1/96 \quad \Pr\{s_0 s_3 s_1 s_3\} = 1/16 \quad \Pr\{s_0 s_3 s_3 s_3\} = 3/64$$

To sum up, the probability is $41/192$, which equals $P_{0,3}^3 = 41/192$.

- b) We define $r_{s,s'}(n)$ to be the probability to first visit s' starting from s , that is (to have independent events)
- $$r_{s,s'}(n) = \Pr\{X(n) = s', X(n-1) \neq s', \dots, X(1) \neq s' \mid X(0) = s\}.$$

The probability of going from s_0 to s_3 in at most 3 steps is $r_{s_0,s_3}(1) + r_{s_0,s_3}(2) + r_{s_0,s_3}(3)$, where

$$\begin{aligned} r_{s_0,s_3}(1) &= \Pr\{s_0 s_3\} = 3/4 \\ r_{s_0,s_3}(2) &= \Pr\{s_0 s_1 s_3\} = 1/24 \quad \text{--- } \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24} \\ r_{s_0,s_3}(3) &= \Pr\{s_0 s_1 s_0 s_3\} = 9/96 \end{aligned}$$

So the probability of going from s_0 to s_3 in at most 3 steps is $3/4 + 1/24 + 9/96 = 85/96$.

c) The initial distribution is $(1/4, 1/4, 1/4, 1/4)$. Since

$$(1/4, 1/4, 1/4, 1/4)P^3 = (17/192, 47/384, 737/1152, 43/288),$$

the probability of being in state s_2 after 3 steps is the last but one entry, $737/1152$.