

Series 4 — Solution

Exercise 1

" $\Rightarrow$ " Assume $P_{=1}(\Diamond a)$ holds in s .

Since $s \models P_{=1}(\Diamond a)$, it holds that we only have to consider the sub-DTMC ^{D'} of all states reachable from s .

By the Long-Run-Theorem it holds for every BSCC B of D' that ~~either~~ all probability mass is in BSCCs

- either B contains an a -state, or
- B can only be reached via an a -state

Also along every τa -path an a -state is reachable, otherwise there was a " τa -BSCC" reachable via an τa -path.

So for every path it holds that it ~~either~~ either reaches an a -state, or it does not but an a -state is always reachable.

$$\Rightarrow \forall ((\exists \Diamond a) \wedge a)$$

" $\Leftarrow$ " Assume $S \models \forall ((\exists \Diamond a) \supset a)$.

$$\Rightarrow S \models \neg \exists (\neg a \cup (\neg a \wedge \neg \exists \Diamond a)) \wedge \neg \exists \Box \neg a$$

$$\Rightarrow S \models \neg \exists \Box \neg a$$

$$\Leftrightarrow S \models \forall \Diamond a$$

$$\Rightarrow \Pr \{ \pi \in \text{Paths}(s) \mid \pi \models \Diamond a \} = 1$$

Exercise 2

Proof by induction over all PCTL-formulae:

S.B.: • $\neg \text{true} \equiv \text{false}$ in NNF ✓

• $\neg \alpha$ in NNF ✓

S.H.: Φ, Ψ can be transformed into NNF.

S.S.: • $\neg(\Phi \wedge \Psi) \equiv (\neg\Phi) \vee (\neg\Psi)$ in NNF

$\underbrace{\quad\quad\quad}_{\text{S.H.}} \quad \underbrace{\quad\quad\quad}_{\text{S.H.}} \quad \Rightarrow$

• $\neg(\neg\Phi) = \Phi$ in NNF

$\underbrace{\quad\quad\quad}_{\text{S.H.}} \quad \Rightarrow$

• $\neg(P_S(\varphi)) \equiv \begin{cases} P_M(\varphi) \vee P_N(\varphi), & \text{if } M \neq \emptyset \neq N, \\ & [0,1] \setminus S = M \dot{\cup} N \\ P_M(\varphi), & \text{if } M \dot{\cup} S = [0,1] \end{cases}$

□

Exercise 3

" $\Rightarrow$ " Suppose $s \sim_{\text{ev}} t$.

Take any $C \in S/\sim_{\text{ev}}$, any $\alpha \in Act$

Case 1: $P(s, \alpha, C) = 1$

$$\Leftrightarrow \exists s' \in C : s \xrightarrow{\alpha} s'$$

$$\begin{array}{l} s \sim_{\text{ev}} t \\ \Rightarrow \end{array} \exists s' \in C : \exists t' : t \xrightarrow{\alpha} t' \wedge s' \sim_{\text{ev}} t'$$

$$\begin{array}{l} C \in S/\sim_{\text{ev}} \\ \Rightarrow \end{array} \exists s' \in C : \exists t' \in C : t \xrightarrow{\alpha} t'$$

$$\Rightarrow \exists t' \in C : t \xrightarrow{\alpha} t'$$

$$\Leftrightarrow P(t, \alpha, C) = 1$$

Case 2: $P(s, \alpha, C) = 0$

$$\Leftrightarrow \neg (\exists s' \in C : s \xrightarrow{\alpha} s')$$

$$\Rightarrow \dots \Rightarrow \neg (\exists t' \in C : t \xrightarrow{\alpha} t')$$

$$\Rightarrow P(t, \alpha, C) = 0$$

$$\Rightarrow s \sim_{\text{ac}} t$$

" $\Leftarrow$ " Suppose $s \sim_{\alpha} t$, and $s \xrightarrow{\alpha} s'$.

$$s \xrightarrow{\alpha} s'$$

$\sim_{\alpha}$ is equiv.
 $\Rightarrow$

$$\exists ! c \in S / \sim_{\alpha} : s \xrightarrow{\alpha} s' \in c$$

$$s \sim_{\alpha} t \stackrel{?}{\Leftarrow} \exists ! c \in S / \sim_{\alpha} : s \xrightarrow{\alpha} s' \in c \quad \sqrt{\exists t' : t \xrightarrow{\alpha} t' \rightarrow t' \in c}$$

$$\Rightarrow [s \xrightarrow{\alpha} s'] \Rightarrow [\exists t' : t \xrightarrow{\alpha} t' \wedge s' \sim_{\alpha} t']$$

Analogously (since $\sim_{\alpha}$ is an equiv. and therefore symmetric) it follows that:

$$[t \xrightarrow{\alpha} t'] \Rightarrow [\exists t' : t \xrightarrow{\alpha} t' \wedge s' \sim_{\alpha} t']$$

$$\Rightarrow s \sim_{\alpha} t$$

Exercise 4

