

Introduction to Model Checking 2015:

Exercise 3. Solution

Theorem that we will repeatedly use: $\{a \mid f(a)\} \subseteq \{a \mid g(a)\}$ iff $\forall x : f(x) \rightarrow g(x)$.

Solution 1

1. $\text{cl}(\emptyset) = \emptyset$.

Proof: Let $t \in \text{cl}(\emptyset)$.

$$\implies \forall x \prec t, \exists t' \in \emptyset : x \prec t'$$

$$\implies \exists t' \in \emptyset$$

$$\implies \text{false}$$

Hence, $t \neq \text{cl}(\emptyset)$. *q.e.d.*

2. $A \subseteq B \implies \text{cl}(A) \subseteq \text{cl}(B)$.

Proof:

$$\begin{aligned} \text{cl}(A) &= \{ t \mid \forall x \prec t, \exists t' \in A : x \prec t' \} \\ &\subseteq \{ t \mid \forall x \prec t, \exists t' \in B : x \prec t' \} \\ &= \text{cl}(B) \quad \text{q.e.d.} \end{aligned}$$

3. $T \subseteq \text{cl}(T)$.

Proof:

$$\begin{aligned} T &= \{ t \mid \exists t' \in T : t = t' \} \\ &= \{ t \mid \forall x \prec t \exists t' : t' \in T \text{ and } t = t' \text{ and } x \prec t' \} \\ &\subseteq \{ t \mid \forall x \prec t \exists t' : t' \in T \text{ and } x \prec t' \} \\ &= \text{cl}(T) \quad \text{q.e.d.} \end{aligned}$$

4. $\text{cl}(\text{cl}(T)) = \text{cl}(T)$.

Proof: It suffices to show $\text{cl}(\text{cl}(T)) \subseteq \text{cl}(T)$. The rest follows from 1.3.

$$\begin{aligned} \text{cl}(\text{cl}(T)) &= \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge (t' \in \text{cl}(T)) \} \\ &= \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge (\forall y : (y \prec t') \rightarrow \exists t'' : (t'' \in T) \wedge y \prec t'') \} \\ &\subseteq \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge ((x \prec t') \rightarrow \exists t'' : (t'' \in T) \wedge x \prec t'') \} \\ &= \{ t \mid \forall x : (x \prec t) \rightarrow (\exists t'' : (t'' \in T) \wedge x \prec t'') \} \\ &= \text{cl}(T) \quad \text{q.e.d.} \end{aligned}$$

5. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

Proof:

$$A \subseteq A \cup B$$

$$\text{cl}(A) \subseteq \text{cl}(A \cup B) \quad \because 1.2.$$

$$\text{cl}(B) \subseteq \text{cl}(A \cup B) \quad \because 1.2.$$

$$\text{cl}(A) \cup \text{cl}(B) \subseteq \text{cl}(A \cup B)$$

Next we mention a small lemma: Let t be any arbitrary but fix infinite word t . For sets $X \subseteq \Sigma^\omega$, $X_t = \{ x \mid \exists t' \in X : x \prec t' \}$. Since the set X_t is downward closed (with resp. $\prec$), it readily follows for any two arbitrary sets A, B :

$$(A \cup B)_t = A_t \text{ or } (A \cup B)_t = B_t$$

$$\begin{aligned} \text{cl}(A \cup B) &= \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge (t' \in A \cup B) \} \\ &\supseteq \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge (t' \in A) \wedge ((A \cup B)_t = A_t) \} \\ &\quad \cup \{ t \mid \forall x : (x \prec t) \rightarrow \exists t' : (x \prec t') \wedge (t' \in B) \wedge ((A \cup B)_t = B_t) \} \\ &= \text{cl}(A) \cup \text{cl}(B) \end{aligned}$$

Solution 2 1. $\text{cl}(\{t\}) = \{t\}$.

Proof: Let $T = \{t\}$.

$$\begin{aligned}\text{cl}(T) &= \{t' \mid \forall x \prec t' \exists t'' \in T : x \prec t''\} \\ &= \{t' \mid \forall x \prec t' : x \prec t\} \\ &= \{t\} \quad \text{q.e.d}\end{aligned}$$

2. $R_a = \{t \in \Sigma^\omega \mid \exists i \in \mathbb{N} : t[i] = a\}$. Consider a word t such that for no i $t[i] = a$. It follows from construction that $t \in \text{cl}(R_a)$. Thus, $R_a \neq \text{cl}(R_a)$.

Choose any arbitrary $t \in \Sigma^\omega$. There can be two cases, $\exists i : t[i] = a$ or for all $\forall i : t[i] \neq a$. For both cases $t \in \text{cl}(R_a)$. Thus $\Sigma^\omega \subseteq \text{cl}(R_a)$.

3.

$$\begin{aligned}\text{cl}(A \cup \overline{\text{cl}(A)}) &= \text{cl}(A) \cup \overline{\text{cl}(\overline{\text{cl}(A)})} \quad \because 1.5 \\ &\subseteq \text{cl}(A) \cup \overline{\text{cl}(A)} \quad \because 1.3 \\ &= \Sigma^\omega\end{aligned}$$

Solution 3 1. The empty set $\emptyset$ is not live. $\therefore \text{cl}(\emptyset) = \emptyset$ and not Σ^ω , from 1.1. Let $A = (\Sigma^+)a^\omega$ and $B = (\Sigma^+)b^\omega$, where $a \neq b$ are symbols. $\text{cl}(A) = \text{cl}(B) = \Sigma^\omega$ but $A \cap B = \emptyset$.

2. $R = \{t \in \Sigma^\omega \mid \exists i \in \mathbb{N} : t[i] = a\}$, where $a \in \Sigma$.

3. Let $R_i = \{t \in \Sigma^\omega \mid t[i] = a\}$. Choose an arbitrary $t \in \Sigma^\omega$ such that $t[i] \neq a$. Then for the prefix $x \prec t$ such that $|x| \geq i$, there exists no $t' \in R_i$ such that $x \prec t'$. Thus $t \notin \text{cl}(R_i)$. Or, $R_i = \text{cl}(R_i)$. This shows that for every $i \in \omega$, R_i is safe. Furthermore, $R = \bigcup_{i \in \omega} R_i$ and R is not safe. In other words:

$$\text{cl}\left(\bigcup_{i \in \omega} R_i\right) = \text{cl}(R) \neq R = \bigcup_{i \in \omega} R_i$$