

Exercise Sheet 4: Denotational Semantics

Due date: May 12th. You can hand in your solutions at the start of the exercise class.

Exercise 1 (Chain Complete Partial Orders)

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Determine whether each of the following statements is true or false. For true statements present a formal proof, and for false statements provide a counterexample.

- (a) [7.5%] Every continuous function $f: (D_1, \sqsubseteq_1) \rightarrow (D_2, \sqsubseteq_2)$ between two CCPOs $(D_1, \sqsubseteq_1)$ and $(D_2, \sqsubseteq_2)$ is monotonic.
- (b) [7.5%] Consider the partial order $(\mathbb{Q}, \leq)$ of the rational numbers ordered by the natural order in the reals. $(\mathbb{Q}, \leq)$ is chain complete.
- (c) [7.5%] If $f: (D_1, \sqsubseteq_1) \rightarrow (D_2, \sqsubseteq_2)$ is a monotonic function between two CCPOs and $D \subseteq D_1$ is a chain, then $f(\bigsqcup D) \sqsubseteq_2 \bigsqcup f(D)$.
- (d) [7.5%] Let $(D, \sqsubseteq)$ be a partial order and let $f: (D, \sqsubseteq) \rightarrow (D, \sqsubseteq)$ be monotonic. If p is the least element in D satisfying $f(p) \sqsubseteq p$, then p is a fixed point of f .

Exercise 2 (repeat-until Loops)

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- (a) [10%] Define a transformer $F: (\Sigma \dashrightarrow \Sigma) \rightarrow (\Sigma \dashrightarrow \Sigma)$ such that

$$\mathcal{C}[\text{repeat } c \text{ until } b] = \text{fix}(F) .$$

The transformer F is allowed to depend on the semantics only of c and b (i.e. $\mathfrak{B}[b]$ and $\mathcal{C}[c]$). You cannot rely on the existence of **while**-loops within the language to define F .

- (b) [5%] Use the definition provided in (a) to compute the transformer $\hat{F}: (\Sigma \dashrightarrow \Sigma) \rightarrow (\Sigma \dashrightarrow \Sigma)$ whose least fixed point gives the semantics of program **repeat skip until false**. In other words, compute $\hat{F}$ such that

$$\mathcal{C}[\text{repeat skip until false}] = \text{fix}(\hat{F}) .$$

- (c) [10%] Show that $\text{fix}(\hat{F}) = f_\emptyset$.

Exercise 3 (Closed Sets)

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A set $C \subseteq D$ is *closed* if and only if for each chain $G \subseteq C$, $\bigsqcup G \in C$. In the following, let $(D, \sqsubseteq)$ be a chain complete partial order and $f: D \rightarrow D$ be a continuous function. Prove the following two statements.

- (a) [7.5%] For each closed set $C \subseteq D$ with $f(x) \in C$ for each $x \in C$, we have $\text{fix}(f) \in C$.

Solution: Let $C \subseteq D$ be a closed set and $f(x) \in C$ for each $x \in C$. By the Tarski-Knaster Theorem,

$$\text{fix}(f) = \bigsqcup \{f^n(\bigsqcup \emptyset) \mid n \geq 0\}.$$

Since $\emptyset$ is a chain and $\emptyset \subseteq C$, $\bigsqcup \emptyset \in C$. By definition, $f(\bigsqcup \emptyset) \in C$ and by induction $f^n(\bigsqcup \emptyset) \in C$ for each $n \geq 0$. Hence, $G = \{f^n(\bigsqcup \emptyset) \mid n \geq 0\} \subseteq C$ is a chain and by definition of closed sets, $\text{fix}(f) = \bigsqcup G \in C$.

(b) [7.5%] $f(x) \sqsubseteq x$ implies $\text{fix}(f) \sqsubseteq x$, $x \in D$.

Solution: Let $C := \{y \in D \mid y \sqsubseteq x\}$. Obviously, C is closed. If $y \in C$ then $f(y) \sqsubseteq f(x) \sqsubseteq x$ and thus $f(y) \in C$. Thus, by (a), $\text{fix}(f) \in C$. By definition of C , $\text{fix}(f) \sqsubseteq x$.

Exercise 4 (Pointwise Ordering)

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Let $(D, \sqsubseteq)$ be a CCPO and define $(D \rightarrow D, \sqsubseteq')$ by setting

$$f_1 \sqsubseteq' f_2 \text{ if and only if } f_1(d) \sqsubseteq f_2(d) \text{ for all } d \in D.$$

(a) [10%] Show that $(D \rightarrow D, \sqsubseteq')$ is a CCPO.

Solution: We have to show that $(D \rightarrow D, \sqsubseteq')$ is a partial order and that each of its chains has a least upper bound. In the following, let $f, f_1, f_2, f_3 \in (D \rightarrow D)$.

1. Reflexivity: By definition, $f \sqsubseteq' f$ if and only if for all $d \in D$, $f(d) \sqsubseteq f(d)$ which is trivially satisfied, because $(D, \sqsubseteq)$ is reflexive.
2. Antisymmetry: By definition, $f_1 \sqsubseteq' f_2$ if and only if for all $d \in D$, $f_1(d) \sqsubseteq f_2(d)$. Since $(D, \sqsubseteq)$ is antisymmetric, it follows $f_2(d) \not\sqsubseteq f_1(d)$ for all $d \in D$ and thus $f_2 \not\sqsubseteq' f_1$.
3. Transitivity: Assume $f_1 \sqsubseteq' f_2 \sqsubseteq' f_3$. By definition, for all $d \in D$, $f_1(d) \sqsubseteq f_2(d) \sqsubseteq f_3(d)$ and since $(D, \sqsubseteq)$ is transitive, $f_1 \sqsubseteq' f_3$.
4. Chain completeness: Let $F \subseteq D \rightarrow D$ be chain. For each $d \in D$, set $F_d := \{f(d) \mid f \in F\}$. Since F is a chain and $(D, \sqsubseteq)$ is a CCPO, F_d is a chain and $\bigsqcup F_d \in D$ exists. By definition, we have $f(d) \sqsubseteq \bigsqcup F_d$ for all $d \in D$ and $f \in F$. Hence, $g : D \rightarrow D, d \mapsto \bigsqcup F_d$ is a least upper bound of F and $g \in D \rightarrow D$.

(b) [20%] Show that fixpoints of chains are “continuous”, i.e.

$$\text{fix}(\bigsqcup' \mathcal{F}) = \bigsqcup \{\text{fix}(f) \mid f \in \mathcal{F}\}$$

holds for all non-empty chains $\mathcal{F} \subseteq D \rightarrow D$ of continuous functions.

Solution: Let $g := \bigsqcup' \mathcal{F}$ and $P := \{\text{fix}(f) \mid f \in \mathcal{F}\}$ and $p := \bigsqcup P$.

“ $\sqsubseteq$ ” We first show $g(p) \sqsubseteq p$. By exercise 3.b, this implies $\text{fix}(g) \sqsubseteq p$.

$$\begin{aligned}
g(p) &= \bigsqcup \{f'(p) \mid f' \in \mathcal{F}\} \\
&= \bigsqcup \{f'(\bigsqcup \{\text{fix}(f) \mid f \in \mathcal{F}\}) \mid f' \in \mathcal{F}\} \\
&= \bigsqcup \{\bigsqcup \{f'(\text{fix}(f)) \mid f \in \mathcal{F}\} \mid f' \in \mathcal{F}\} \\
&\stackrel{(*)}{\sqsubseteq} \bigsqcup \{\bigsqcup \{\text{fix}(f) \mid f \in \mathcal{F}\} \mid f' \in \mathcal{F}\} \\
&= \bigsqcup \{\text{fix}(f) \mid f \in \mathcal{F}\} \\
&= p
\end{aligned}$$

Why does $(*)$ hold? By definition, $\mathcal{F}$ is a chain, i.e. $f \sqsubseteq f'$ or $f' \sqsubseteq f$.

1. If $f \sqsubseteq f'$ then $f'(\text{fix}(f)) = f'(\bigsqcup \{f^n(\perp) \mid n \in \mathbb{N}\}) \sqsubseteq f'(\bigsqcup \{f'^n(\perp) \mid n \in \mathbb{N}\}) = \text{fix}(f')$
2. If $f' \sqsubseteq f$ then $f'(\text{fix}(f)) \sqsubseteq f(\text{fix}(f)) = \text{fix}(f)$

Thus for each pair $f, f' \in \mathcal{F}$ there exists a $f'' \in \{f, f'\}$ such that $f'(\text{fix}(f)) \sqsubseteq \text{fix}(f'')$. Then, every upper bound of $\{\text{fix}(f) \mid f \in \mathcal{F}\}$ is also an upper bound of $\{f'(\text{fix}(f)) \mid f \in \mathcal{F}\}$ regardless of the choice of $f' \in \mathcal{F}$. In particular, $\bigsqcup \{f'(\text{fix}(f)) \mid f \in \mathcal{F}\} \sqsubseteq \bigsqcup \{\text{fix}(f) \mid f \in \mathcal{F}\}$ for each $f' \in \mathcal{F}$.

“ $\supseteq$ ” The converse direction is simple. Since $f \sqsubseteq g$ and f, g are monotone, $f^n \sqsubseteq g^n$ for each $n \geq 0$. Hence, $\text{fix}(f) = \bigsqcup \{f^n(\bigsqcup \emptyset) \mid n \geq 0\} \sqsubseteq \bigsqcup \{g^n(\bigsqcup \emptyset) \mid n \geq 0\} = \text{fix}(g)$ for each $f \in \mathcal{F}$. Thus, $\text{fix}(g)$ is an upper bound of P and therefore $p \sqsubseteq \text{fix}(g)$.