

Exercise 1 (Equivalences):

(4 points)

Let φ, ψ, π be arbitrary LTL formulae. For each of the following pairs of LTL formulae, determine in which relation they are. More specifically, determine whether they are equivalent, one of them subsumes the other or they are incomparable. Prove your claims.

- a) $\Diamond\Box\varphi$ and $\Box\Diamond\varphi$
- b) $\Diamond\Box\varphi \wedge \Diamond\Box\psi$ and $\Diamond(\Box\varphi \wedge \Box\psi)$
- c) $\varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$ and $\Box\Diamond\varphi$
- d) $(\varphi U \psi) U \pi$ and $\varphi U (\psi U \pi)$

Solution: _____

- a) The two formulae are not equivalent. Take $\varphi = a$, then $(\{a\}^\omega) \models \Box\Diamond\varphi$, but $(\{a\}^\omega) \not\models \Diamond\Box\varphi$. However, $\Diamond\Box\varphi$ implies $\Box\Diamond\varphi$.

Proof.

$$\begin{aligned}
 & A_0 A_1 \dots \models \Diamond\Box\varphi \\
 \Rightarrow & \exists j \geq 0 \ A_j A_{j+1} \dots \models \Box\varphi \\
 \Rightarrow & \exists j \geq 0 \ \forall k \geq j \ A_k A_{k+1} \dots \models \varphi \\
 \Rightarrow & \exists j \geq 0 \ \forall k \geq j \ \exists \ell \geq k \ A_\ell A_{\ell+1} \dots \models \varphi \\
 \Rightarrow & \exists j \geq 0 \ \forall k \geq 0 \ \exists \ell \geq k \ A_\ell A_{\ell+1} \dots \models \varphi \\
 \Rightarrow & \forall k \geq 0 \ \exists \ell \geq k \ A_\ell A_{\ell+1} \dots \models \varphi \\
 \Rightarrow & A_0 A_1 \dots \models \Box\Diamond\varphi
 \end{aligned}$$

- b) The two formulae are indeed equivalent.

Proof.

$$\begin{aligned}
 & A_0 A_1 \dots \models \Diamond(\Box\varphi \wedge \Box\psi) \\
 \Leftrightarrow & \exists j \geq 0 \ A_j A_{j+1} \dots \models \Box\varphi \wedge \Box\psi \\
 \Leftrightarrow & \exists j \geq 0 \ (\forall k \geq j \ A_k A_{k+1} \dots \models \varphi \text{ and } \forall k \geq j \ A_k A_{k+1} \dots \models \psi) \\
 \stackrel{(*)}{\Leftrightarrow} & \exists j_1 \geq 0 \ \forall k \geq j_1 \ A_k A_{k+1} \dots \models \varphi \text{ and } \exists j_2 \geq 0 \ \forall k \geq j_2 \ A_k A_{k+1} \dots \models \psi \\
 \Leftrightarrow & \exists j_1 \geq 0 \ A_{j_1} A_{j_1+1} \dots \models \Box\varphi \text{ and } \exists j_2 \geq 0 \ A_{j_2} A_{j_2+1} \dots \models \Box\psi \\
 \Leftrightarrow & A_0 A_1 \dots \models \Diamond\Box\varphi \text{ and } A_0 A_1 \dots \models \Diamond\Box\psi \\
 \Leftrightarrow & A_0 A_1 \dots \models \Diamond\Box\varphi \wedge \Diamond\Box\psi
 \end{aligned}$$

where for $(*)$ and " $\Leftarrow$ " we take $j := \max\{j_1, j_2\}$, and for " $\Rightarrow$ " we take $j_1, j_2 := j$.

- c) The two formulae are not equivalent. Take $\varphi = a$, then $\emptyset\{a\}^\omega \models \Box\Diamond\varphi$, but $\emptyset\{a\}^\omega \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$. However, $A_0 A_1 \dots \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$ implies $A_0 A_1 \dots \models \Box\Diamond\varphi$.

Proof. We show that $A_0A_1 \dots \not\models \Box\Diamond\varphi$ implies $A_0A_1 \dots \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$. Therefore assume $A_0A_1 \dots \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$ for the purpose of a contradiction. Then

$$\begin{aligned} A_0A_1 \dots &\not\models \Box\Diamond\varphi \\ A_0A_1 \dots &\models \neg\Box\Diamond\varphi \\ \Rightarrow A_0A_1 \dots &\models \Diamond\neg\Diamond\varphi \\ \Rightarrow \exists j^* \geq 0 \forall k \geq j^* &A_kA_{k+1} \dots \models \neg\varphi \end{aligned}$$

- Case 1: there exists no $j \geq 0$ with $A_jA_{j+1} \dots \models \varphi$, then $A_0A_1 \dots \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$ and nothing remains to show.
- Case 2: there exists $j \geq 0$ with $A_jA_{j+1} \dots \models \varphi$ and we conclude $j < j^*$. Let $j^\uparrow < j^*$ be the highest such index.

$$\begin{aligned} A_0A_1 \dots &\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi) \\ \Rightarrow A_0A_1 \dots &\models \varphi \text{ and } \forall j \geq 0 \ A_jA_{j+1} \dots \models \varphi \rightarrow \bigcirc\Diamond\varphi \\ \Rightarrow \forall j \geq 0 \ (A_jA_{j+1} \dots \models \varphi &\Rightarrow A_jA_{j+1} \dots \models \bigcirc\Diamond\varphi) \end{aligned}$$

Hence, we have that

(i) $A_0A_1 \dots \models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$ and

(ii) $A_{j^\uparrow}A_{j^\uparrow+1} \dots \models \varphi$

imply $A_{j^\uparrow}A_{j^\uparrow+1} \dots \models \bigcirc\Diamond\varphi$. Then

$$\begin{aligned} A_{j^\uparrow}A_{j^\uparrow+1} \dots &\models \bigcirc\Diamond\varphi \\ \Rightarrow A_{j^\uparrow+1}A_{j^\uparrow+2} \dots &\models \Diamond\varphi \\ \Rightarrow \exists j > j^\uparrow \ A_jA_{j+1} \dots &\models \varphi \end{aligned}$$

which contradicts that $j^\uparrow$ is the largest index such that $A_{j^\uparrow}A_{j^\uparrow+1} \dots \models \varphi$. We can therefore conclude $A_0A_1 \dots \not\models \varphi \wedge \Box(\varphi \rightarrow \bigcirc\Diamond\varphi)$.

d) The two formulae are incomparable in the sense that there are two words α_1 and α_2 such that

- $\alpha_1 \not\models (\varphi \cup \psi) \cup \pi$ but $\alpha_1 \models \varphi \cup (\psi \cup \pi)$
- $\alpha_2 \models (\varphi \cup \psi) \cup \pi$ but $\alpha_2 \not\models \varphi \cup (\psi \cup \pi)$

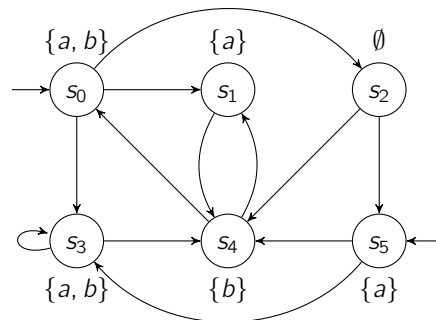
Take $\varphi = a$, $\psi = b$ and $\pi = c$. Suitable counterexamples then are

- $\alpha_1 = \{a\}\{c\}^\omega$ and
- $\alpha_2 = \{a\}\{b\}\{a\}\{b\}\{c\}^\omega$

Exercise 2 (LTL satisfaction on TS):

(2 points)

Consider the following transition system TS:



Determine whether $TS \models \varphi_i$ for each of the following properties. Justify your answers.

- a) $\varphi_1 = \Box \Diamond a$
- b) $\varphi_2 = \Diamond \Box a$
- c) $\varphi_3 = a \rightarrow \bigcirc \bigcirc a$
- d) $\varphi_4 = bRa$ where $\varphi R \psi \stackrel{\text{def}}{=} \neg(\neg\varphi U \neg\psi)$

Solution: _____

- a) $TS \models \varphi_1$, since there is no non-trivial strongly-connected set of states (note: this set does not necessarily have to be maximal) without an a -labeled state in TS .
- b) $TS \not\models \varphi_2$, since the path $(s_0 s_2 s_4)^\omega$ exhibits the trace $(\{a, b\} \emptyset \{b\})^\omega \not\models \Diamond \Box a$.
- c) $TS \not\models \varphi_3$, since the path $(s_0 s_1 s_4)^\omega$ exhibits the trace $(\{a, b\} \{a\} \{b\})^\omega \not\models a \rightarrow \bigcirc \bigcirc a$.
- d) Note that bRa can be interpreted as “ b releases a ”, i.e. a has to hold forever, unless at some point a and b hold in which case the requirement on a is released. With this in mind, we have $TS \not\models \varphi_4$, since the path $s_5(s_4 s_1)^\omega$ exhibits the trace $\{a\}(\{b\}\{a\})^\omega \not\models bRa$.

Exercise 3 (New LTL Operators):

(2 points)

Let φ and ψ be LTL formulae and $i \in \mathbb{N}$. Consider the following new operators:

- (i) “surround”: $\varphi \mathcal{S} \psi$ holds if for any position i at which ψ holds, φ holds sometime before and sometime after i (every occurrence of ψ is “surrounded” by occurrences of φ).
 - (ii) “jump”: $\mathcal{J}^i \varphi$ is true if, starting with the current position, φ is true at every i -th step (for example $\mathcal{J}^2 a$ states that a has to hold at every even position).
- a) Formalize the semantics of these operators.
 - b) For each of the two new operators argue whether or not they can be expressed in terms of regular LTL.

Solution: _____

- a) Let $\sigma = A_0 A_1 \dots \in (2^{\text{AP}})^\omega$. Then

$$\begin{aligned} \sigma \models \varphi \mathcal{S} \psi \text{ iff } & \forall j \geq 0 (A_j A_{j+1} \dots \models \psi \implies \exists j_1 < j \exists j_2 > j A_{j_1} A_{j_1+1} \dots \models \varphi \text{ and } A_{j_2} A_{j_2+1} \dots \models \psi) \\ \sigma \models \mathcal{J}^i \varphi \text{ iff } & \forall j \geq 0 A_{j \cdot i} A_{j \cdot i + 1} \dots \models \varphi \end{aligned}$$

- b) The “surrounds” operator can be expressed in regular LTL, because we have

$$\varphi \mathcal{S} \psi \equiv \underbrace{\neg(\neg\varphi U \psi)}_{(I)} \wedge \underbrace{\Box(\psi \rightarrow \bigcirc \Diamond \varphi)}_{(II)}$$

where

- (I) means φ holds before any ψ holds (note that this does also rule out the cases where φ and ψ become true at the very same time) and

(II) means that if at some point ψ holds, at a later position φ holds again.

The “jump” operator on the other hand, cannot be expressed in terms of regular LTL. As stated in the exercise $\mathcal{J}^2 a$ means that a has to hold at every even position in an infinite word. This is a counting language and, intuitively speaking, LTL has no means to “count”. Therefore, the language is not definable in LTL. Note that, for example, the formula

$$a \leftrightarrow \bigcirc \bigcirc a$$

is *not* defining the same set of words as $\mathcal{J}^2 a$.

Exercise 4 (Positive Normal Form):

(2 points)

Provide a sequence (φ_n) of LTL formulae such that the LTL formulae ψ_n is in PNF (including weak-until), $\varphi_n \equiv \psi_n$, and ψ_n is exponentially longer than φ_n . Use the transformation rules from the lecture.

Solution: _____

Consider the sequence $(\varphi_n) = (\neg\varphi'_n)$ of formulae where

$$\varphi'_i = \begin{cases} a_0 & \text{if } i = 0 \\ a_i \mathbf{U} \varphi'_{n-1} & \text{if } i > 0 \end{cases}$$

Let $\text{PNF}(\varphi)$ denote the formula obtained by transforming φ into PNF. Then:

$$\varphi_n = \neg\varphi'_n \stackrel{\text{PNF}}{\equiv} \begin{cases} \neg a_0 & n = 0 \\ \neg(\text{PNF}(\neg\varphi'_{n-1}))\mathbf{W}(\neg a_n \wedge \text{PNF}(\neg\varphi'_{n-1})) = \underbrace{(\text{PNF}(\varphi_{n-1}))\mathbf{W}(\neg a_n \wedge \text{PNF}(\varphi_{n-1}))}_{\psi_n} & n > 0 \end{cases}$$

The length of ψ_n can therefore be represented by the recursive equation

$$L'(n) = \begin{cases} 2 & \text{if } n = 0 \\ 2L'(n-1) + 4 & \text{if } n > 0 \end{cases}$$

Let

$$L(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2L(n-1) & \text{if } n > 0 \end{cases}$$

Clearly $L(n) < L'(n)$ for all n . However, $L(n)$ can be put in closed form as

$$L(n) = 2^n$$

(which can be easily proved by induction). On the other hand, the size of φ_n is

$$\underbrace{1}_{\neg} + \underbrace{\sum_{i=1}^n 2}_{\text{every step adds an AP and an "until"}} + \underbrace{1}_{a_0} = 2n + 2$$

In summary, while the size of φ_n grows linearly in n , the size of ψ_n grows (at least) exponentially.