

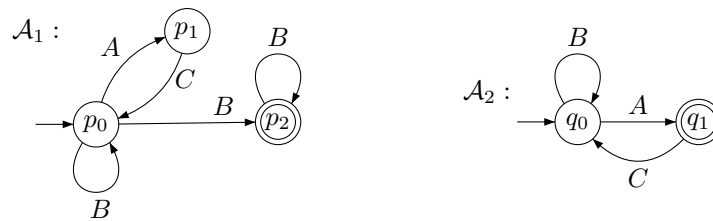
Exercise 1 (GNBA):

(2 points)

- Provide NBA $\mathcal{A}_1$ and $\mathcal{A}_2$ for the languages given by the expressions $(AC + B)^*B^\omega$ and $(B^*AC)^\omega$.
- Apply the product construction to obtain an GNBA $\mathcal{G}$ and an NBA $\mathcal{A}$ with $\mathcal{L}_\omega(\mathcal{A}) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$.
Hint: Do not apply simplifications in these steps
- Justify, why $\mathcal{L}_\omega(\mathcal{G}) = \emptyset$ where $\mathcal{G}$ denotes the GNBA accepting the intersection.

Solution:

NBA $\mathcal{A}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, F_1)$ and $\mathcal{A}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, F_2)$ for the languages:



The corresponding GNBA are given by

$$\mathcal{G}_1 = (Q_1, \Sigma, \delta_1, Q_{0,1}, \{F_1\})$$

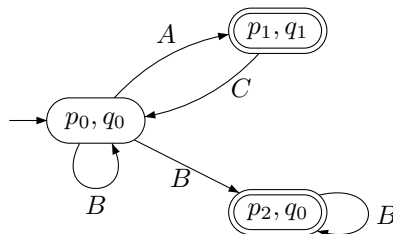
$$\mathcal{G}_2 = (Q_2, \Sigma, \delta_2, Q_{0,2}, \{F_2\})$$

Applying the product construction yields the following GNBA:

$$\mathcal{G} = (Q_1 \times Q_2, \Sigma, \delta, Q_{0,1} \times Q_{0,2}, \mathcal{F}) \text{ where}$$

- $\delta((p, q), A) = \delta_1(p, A) \times \delta_2(q, A)$
- $\mathcal{F} = \{F_1 \times Q_2\} \cup \{Q_1 \times F_2\} = \{(p_2, q_0), (p_2, q_1)\}, \{(p_0, q_1), (p_1, q_1), (p_2, q_1)\}$

The automaton $\mathcal{G}$ can be outlined as follows (only reachable states are outlined below):



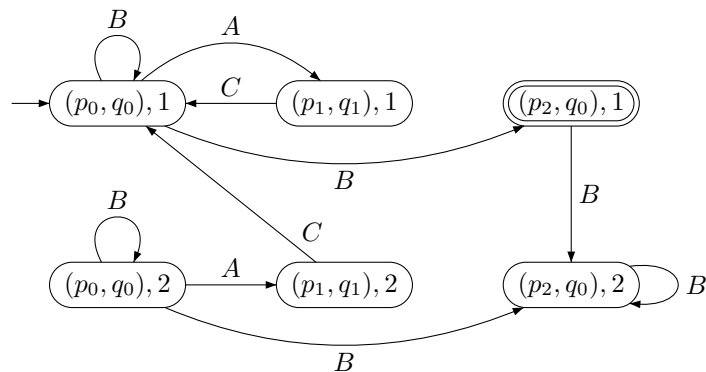
Its acceptance component is $\mathcal{F} = \{(p_2, q_0)\}, \{(p_1, q_1)\}$.

According to the acceptance condition of generalized Büchi automata, $\mathcal{G}$ accepts an input word α if and only if for each $F \in \mathcal{F}$ some states are visited infinitely often. But as soon as (p_2, q_0) is visited, F_1 is not reachable any

longer.

Therefore $\mathcal{G}$ only accepts the empty language.

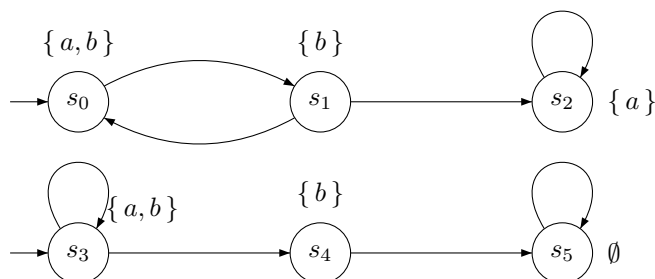
Given $\mathcal{G}$, construct an equivalent NBA $\mathcal{A}$:



Again, on each possible run, the state $((p_2, q_0), 2)$ of $\mathcal{A}$ can be visited only once.
Therefore also $\mathcal{L}_\omega(\mathcal{A}) = \emptyset$ holds.

Exercise 2 (LTL and Fairness):

(3 points)



Consider the transition system TS above with the set $AP = \{a, b, c\}$ of atomic propositions. Note that this is a single transition system with two initial states. Consider the LTL fairness assumption

$$fair = (\Box\Diamond(a \ \& \ b) \rightarrow \Box\Diamond\neg c) \ \wedge \ (\Diamond\Box(a \ \& \ b) \rightarrow \Box\Diamond\neg b).$$

- Determine the fair paths in TS, i.e., the initial, infinite paths satisfying *fair*
- For each of the following LTL formulae:

$$\begin{aligned} \varphi_1 &= bU\Box\neg b \\ \varphi_2 &= bW\Box\neg b \\ \varphi_3 &= \bigcirc \bigcirc bU\Box\neg b \end{aligned}$$

determine whether $TS \models_{fair} \varphi_i$. In case $TS \not\models_{fair} \varphi_i$, indicate a path $\pi \in Paths(TS)$ for which $\pi \not\models \varphi_i$.

- Redo the previous task but ignore the fairness assumption.

Solution: _____

- a) The fair paths of TS are defined by:

$$fair = (\Box\Diamond(a \wedge b) \rightarrow \Box\Diamond\neg c) \wedge (\Diamond\Box(a \wedge b) \rightarrow \Box\Diamond\neg b)$$

The conclusion in the first conjunction $(\Box\Diamond(a \wedge b) \rightarrow \Box\Diamond\neg c)$ is fulfilled by every path, since no state in TS is labeled with c . Formally, this follows from $\Box\neg c \rightarrow \Box\Diamond\neg c$. Consider the second conjunct $(\Diamond\Box(a \wedge b) \rightarrow \Box\Diamond\neg b)$ of *fair*: Its premise is fulfilled only on the path $\pi = s_3^\omega$. But $\pi \not\models \Box\Diamond\neg b$. Therefore π is (the only) unfair path in TS. We conclude that:

$$FairPaths(TS) = \mathcal{L}_\omega((s_0s_1)^\omega + (s_0s_1)^+s_2^\omega + s_3^+s_4s_5^\omega)$$

- b) – $\varphi_1 = bU\Box\neg b$:

Consider the path $\pi_2 = (s_0s_1)^\omega \in FairPaths(TS)$. Here, we have

$$trace(\pi_2) = \sigma_2 = (\{a, b\}\{b\})^\omega$$

and $\sigma_2 \not\models bU\Box\neg b$ since there exists no $i \geq 0$ s.t. $\sigma_2[i..] \models \Box\neg b$.

$\Rightarrow TS \not\models_{fair} bU\Box\neg b$

- $\varphi_2 = bW\Box\neg b$:

It holds $TS \models_{fair} \varphi_2$, since $(s_0s_1)^\omega \models \Box b$, $(s_0s_1)^+s_2^\omega \models bU\Box\neg b$, and $s_3^+s_4s_5^\omega \models bU\Box\neg b$ and $s_3^\omega \models \Box b$.

- $\varphi_3 = ()U\Box\neg b$:

Consider the path $\pi = s_0s_1s_2^\omega \in FairPaths(TS)$. Here, we have

$$trace(\pi) = \sigma = \{a, b\}\{b\}\{a\}^\omega$$

and $\sigma \not\models (()U\Box\neg b$ since $\sigma[0..]$ neither satisfies $\Box\neg b$ nor $()U b$.

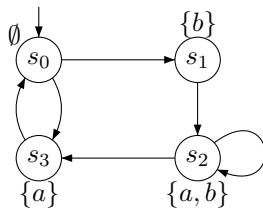
$\Rightarrow TS \not\models_{fair} (()U\Box\neg b$

- c) – By definition of fairness $Traces_{fair}(TS) \subseteq Traces(TS)$. So if $FairTraces(TS) \not\subseteq Words(\varphi)$ then also $Traces(TS) \not\subseteq Words(\varphi)$. From this we immediately conclude that $TS \not\models \varphi_1$ and $TS \not\models \varphi_3$.
– If we ignore the fairness assumption, there is one more path to be checked: $\pi = s_3^\omega$. However $\pi \models \Box b \Rightarrow \pi \models \varphi_2$. Hence $TS \models \varphi_2$.

Exercise 3 (Model Checking):

(3 points)

TS:



We consider the LTL formula $\varphi = \Box(a \rightarrow (\neg bU(a \wedge b)))$ over the set $AP = \{a, b\}$ of atomic propositions and want to check $TS \models \varphi$ wrt. the transition system outlined above.

- a) To check $TS \models \varphi$, convert $\neg\varphi$ into an equivalent LTL-formula ψ which is constructed according to the following grammar:

$$\varphi ::= true \mid false \mid a \mid b \mid \varphi \wedge \varphi \mid \neg\varphi \mid \varphi U \varphi.$$

Then construct $closure(\psi)$.

- b) Give the elementary sets wrt. $closure(\psi)$!

- c) Construct the GNBA $\mathcal{G}_\psi$ by providing its initial states, its acceptance set and its transition relation. Use the algorithm given in the lecture.
Hint: It suffices to provide the transition relation as a table.
- d) Now, construct an NBA $\mathcal{A}_{\neg\varphi}$ **directly** from $\neg\varphi$, i.e. without relying on $\mathcal{G}_\psi$.
Hint: Four states suffice!
- e) Construct $TS \otimes \mathcal{A}_{\neg\varphi}$.
- f) Use the Nested DFS algorithm from the lecture to check $TS \models \varphi$. Therefore sketch the algorithm's main steps and interpret its outcome!

Solution: _____

1. Transform the negation $\neg\varphi = \neg\Box(a \rightarrow (\neg bU(a \wedge b)))$ into an equivalent LTL-formula according to the basic LTL-syntax:

$$\begin{aligned}\neg\varphi &= \neg\Box(a \rightarrow (\neg bU(a \wedge b))) \\ &\equiv \Diamond\neg(\neg a \vee (\neg bU(a \wedge b))) \\ &\equiv \Diamond(a \wedge \neg(\neg bU(a \wedge b))) \\ &\equiv \underbrace{trueU(a \wedge \neg(\neg bU(a \wedge b)))}_{\psi}\end{aligned}$$

We then compute the closure:

$$\begin{aligned}closure(\psi) &= \{ true, false, a, \neg a, b, \neg b \\ &\quad a \wedge b, \neg(a \wedge b), \neg bU(a \wedge b), \neg(\neg bU(a \wedge b)), \\ &\quad a \wedge \neg(\neg bU(a \wedge b)), \neg(a \wedge \neg(\neg bU(a \wedge b))), \psi, \neg\psi \}\end{aligned}$$

2. The elementary sets are as follows:

	a	b	$a \wedge b$	$\neg bU(a \wedge b)$	$a \wedge \neg(\neg bU(a \wedge b))$	ψ
B_1	0	0	0	0	0	0
B_2	0	0	0	0	0	1
B_3	0	0	0	1	0	0
B_4	0	0	0	1	0	1
B_5	0	1	0	0	0	0
B_6	0	1	0	0	0	1
B_7	1	0	0	0	1	1
B_8	1	0	0	1	0	0
B_9	1	0	0	1	0	1
B_{10}	1	1	1	1	0	0
B_{11}	1	1	1	1	0	1

3. The elementary sets form the states of the GNBA $\mathcal{G}_{\neg\varphi} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$. We have

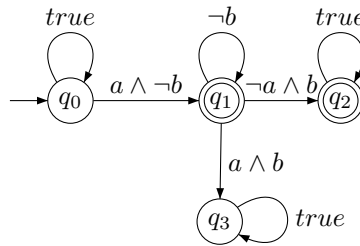
$$\begin{aligned}Q &= \{B_1, \dots, B_{11}\} \\ Q_0 &= \{B_2, B_4, B_6, B_7, B_9, B_{11}\} \\ \mathcal{F} &= \{F_{\neg bU(a \wedge b)}, F_\psi\}, \text{ where} \\ F_{\neg bU(a \wedge b)} &= \{B_1, B_2, B_5, B_6, B_7, B_{10}, B_{11}\} \\ F_\psi &= \{B_1, B_3, B_5, B_7, B_8, B_{10}\}\end{aligned}$$

The transition relation is given as follows:

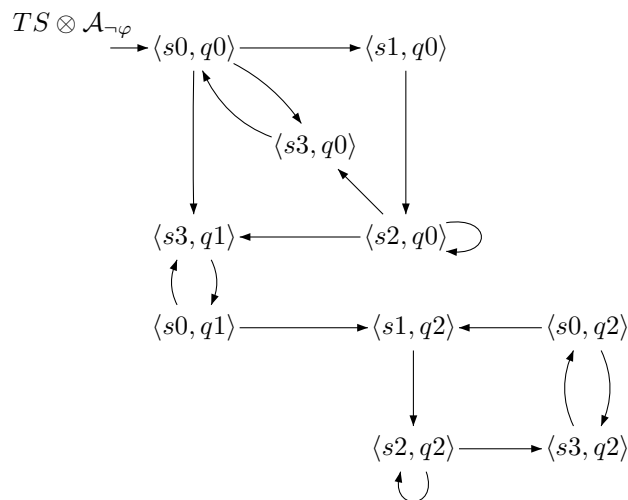
	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
B_1	$\emptyset$				$\emptyset$						
B_2		$\emptyset$				$\emptyset$	$\emptyset$				
B_3			$\emptyset$					$\emptyset$		$\emptyset$	
B_4				$\emptyset$					$\emptyset$		$\emptyset$
B_5	$\{b\}$		$\{b\}$		$\{b\}$			$\{b\}$		$\{b\}$	
B_6		$\{b\}$		$\{b\}$		$\{b\}$	$\{b\}$		$\{b\}$		$\{b\}$
B_7	$\{a\}$	$\{a\}$			$\{a\}$	$\{a\}$	$\{a\}$				
B_8			$\{a\}$					$\{a\}$		$\{a\}$	
B_9				$\{a\}$					$\{a\}$		$\{a\}$
B_{10}	$\{a, b\}$		$\{a, b\}$		$\{a, b\}$			$\{a, b\}$		$\{a, b\}$	
B_{11}		$\{a, b\}$		$\{a, b\}$		$\{a, b\}$	$\{a, b\}$		$\{a, b\}$		$\{a, b\}$

4. The following NBA $\mathcal{A}_{\neg\varphi}$ exactly recognizes $Words(\neg\varphi)$. To see this, consider the following equivalent LTL-formula:

$$\begin{aligned}
 \neg\varphi &\equiv \neg\Box(a \rightarrow (\neg bU(a \wedge b))) \\
 &\equiv \Diamond\neg(\neg a \vee (\neg bU(a \wedge b))) \\
 &\equiv \Diamond(a \wedge \neg(\neg bU(a \wedge b))) \\
 &\equiv \Diamond(a \wedge (bR(\neg a \vee \neg b)))
 \end{aligned}$$



- (e) The product transition system $TS \otimes \mathcal{A}_{\neg\varphi}$ is depicted below:



- (f) We now check the validity of the persistence property “always $\bigwedge_{q \in F} \neg q$ ” where $F = \{q_1, q_2\}$ is the set of accepting states of $\mathcal{A}_{\neg\varphi}$. Therefore we use the nested DFS algorithm. Its outcome is graphically represented below:



Therefore, in our traversal the NDFS algorithm yields the following counterexample:

$$Reverse(V.U) = \langle s_0, q_0 \rangle \langle s_3, q_1 \rangle \langle s_0, q_1 \rangle \langle s_1, q_2 \rangle \langle s_2, q_2 \rangle \langle s_3, q_2 \rangle \langle s_0, q_2 \rangle \langle s_3, q_2 \rangle \langle s_0, q_2 \rangle$$

The corresponding trace $\sigma \in Traces(TS) = \emptyset\{a\}\emptyset\{b\}\{a, b\}(\{a\}\emptyset)^\omega$ indeed violates the formula $\varphi = \Box(a \rightarrow (\neg bU(a \wedge b)))$. *Note:* Dependent on the choice of the successor states during the NDFS, different counterexamples are obtained!