

Exercise 1 (LTL Operators):

(2 points)

Let φ and ψ be LTL formulae. Consider the following new operators:

a) "At next" $\varphi AX \psi$:

$$\begin{aligned} A_1 A_2 \dots \models \varphi AX \psi &\iff \text{for all } i \geq 0 \text{ where } A_i A_{i+1} \dots \models \psi, \\ &\text{for which there exists no } 0 \leq j < i \text{ where } A_j A_{j+1} \dots \models \psi, \\ &A_i A_{i+1} \dots \models \varphi \text{ holds} \end{aligned}$$

b) "While" $\varphi WH \psi$:

$$\begin{aligned} A_1 A_2 \dots \models \varphi WH \psi &\iff \text{for all } i \geq 0 \text{ where } A_j A_{j+1} \dots \models \psi \text{ for all } 0 \leq j < i, \\ &A_k A_{k+1} \dots \models \varphi \text{ for all } 0 \leq k < i \end{aligned}$$

c) "Before" $\varphi B \psi$:

$$\begin{aligned} A_1 A_2 \dots \models \varphi B \psi &\iff \text{for all } i \geq 0 \text{ where } A_i A_{i+1} \dots \models \psi, \\ &\text{there exists some } 0 \leq j < i \text{ where } A_j A_{j+1} \dots \models \varphi \end{aligned}$$

Show that these operators are LTL-definable by providing equivalent LTL formulae. You may use both the until and weak until operator.

Solution: _____

1.

$$\begin{aligned} \sigma &\models \varphi AX \psi \\ &\iff (\forall 0 \leq i. \sigma[i..] \models \psi \wedge \neg(\exists 0 \leq j < i. \sigma[j..] \models \psi)) \Rightarrow \sigma[i..] \models \varphi \\ &\iff (\exists 0 \leq i. \sigma[i..] \models \psi \wedge (\forall 0 \leq j < i. \sigma[j..] \models \neg\psi) \wedge \sigma[i..] \models \varphi) \\ &\quad \vee (\forall 0 \leq i. \sigma[i..] \models \neg\psi) \\ &\iff \sigma \models \neg\psi U(\psi \wedge \varphi) \vee \Box \neg\psi \\ &\iff \sigma \models \neg\psi W(\psi \wedge \varphi) \end{aligned}$$

2.

$$\begin{aligned} \sigma &\models \varphi WH \psi \\ &\iff (\forall 0 \leq i. (\forall 0 \leq j < i. \sigma[i..] \models \psi) \Rightarrow (\forall 0 \leq k < i. \sigma[k..] \models \varphi)) \\ &\iff (\exists 0 \leq i. \sigma[i..] \models \neg\psi \wedge (\forall 0 \leq j < i. \sigma[i..] \models \varphi \wedge \psi)) \\ &\quad \vee (\forall 0 \leq i. \sigma[i..] \models \varphi \wedge \psi) \\ &\iff \sigma \models (\varphi \wedge \psi) U \neg\psi \vee \Box(\varphi \wedge \psi) \\ &\iff \sigma \models (\varphi \wedge \psi) W \neg\psi \end{aligned}$$

3.

$$\begin{aligned}
 & \sigma \models \varphi B \psi \\
 \iff & (\forall 0 \leq i. \sigma[i..] \models \psi \Rightarrow (\exists 0 \leq j < i. \sigma[j..] \models \varphi)) \\
 \iff & (\exists 0 \leq i. \sigma[i..] \models \varphi \wedge (\forall 0 \leq j \leq i. \sigma[j..] \models \neg\psi)) \\
 & \vee (\forall 0 \leq i. \sigma[i..] \models \neg\psi) \\
 \iff & (\exists 0 \leq i. \sigma[i..] \models \varphi \wedge (\forall 0 \leq j < i. \sigma[j..] \models \neg\psi) \wedge (\forall 0 \leq j < i. \sigma[j+1..] \models \neg\psi)) \\
 & \vee (\forall 0 \leq i. \sigma[i..] \models \neg\psi) \\
 \iff & \sigma \models (\neg\psi \wedge \bigcirc \neg\psi) U \varphi \vee \Box \neg\psi \\
 \iff & \sigma \models (\neg\psi \wedge \bigcirc \neg\psi) W \varphi
 \end{aligned}$$

Exercise 2 (LTL to Büchi):

(3 points)

Let $\varphi = (a \wedge \bigcirc a) U (a \wedge \neg \bigcirc a)$ be an LTL-formula over $AP = \{a\}$.

1. Compute all elementary sets with respect to φ .
2. Construct the GNBA $\mathcal{G}_\varphi$ according to the algorithm from the lecture such that $\mathcal{L}_\omega(\mathcal{G}_\varphi) = \text{Words}(\varphi)$.
3. Give an ω -regular expression E such that $\mathcal{L}_\omega(\mathcal{G}_\varphi) = \mathcal{L}_\omega(E)$.

Solution: _____

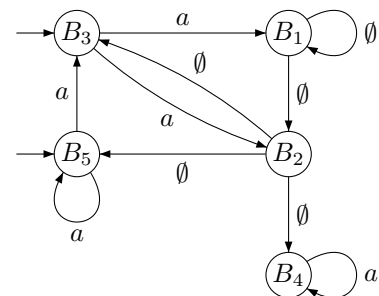
1. The elementary sets are:

	a	$\bigcirc a$	$a \wedge \bigcirc a$	$a \wedge \neg \bigcirc a$	φ
B_1	0	0	0	0	0
B_2	0	1	0	0	0
B_3	1	0	0	1	1
B_4	1	1	1	0	0
B_5	1	1	1	0	1

2. The GNBA $\mathcal{G}_\varphi = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$ is defined by:

$$\begin{aligned}
 Q &= \{B_1, B_2, B_3, B_4, B_5\} \\
 Q_0 &= \{B_3, B_5\} \\
 \mathcal{F} &= \{F_\varphi\} \\
 F_\varphi &= \{B_1, B_2, B_3, B_4\}
 \end{aligned}$$

The transition relation δ is given by the following graph:



3. We derive $\mathcal{L}_\omega(\mathcal{G}_\varphi) = \text{Words}(\varphi) = \{a\}^+ \emptyset (2^{AP})^\omega$.

Exercise 3 (CTL Equivalences):

(3 points)

Prove or disprove the following implications:

- (a) Let $\Phi_1 = \forall \Diamond a \vee \forall \Diamond b$ and $\Phi_2 = \forall \Diamond (a \vee b)$.
Prove or disprove the following implications: $\Phi_1 \implies \Phi_2$ and $\Phi_2 \implies \Phi_1$.
- (b) Now consider $\Psi_1 = \exists (aU\exists(bUc))$ and $\Psi_2 = \exists(\exists(aUb)Uc)$.
Again, prove or disprove $\Psi_1 \implies \Psi_2$ and $\Psi_2 \implies \Psi_1$.

Solution: _____

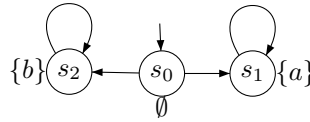
- (a) – We first consider the first implication, $\Phi_1 \implies \Phi_2$. We prove for any transition system TS , that $s \models \Phi_1 \implies s \models \Phi_2$:

$$s \models \Phi_1 \iff s \models \forall \Diamond a \text{ or } s \models \forall \Diamond b$$

$$\begin{aligned} \text{Case 1: } s \models \forall \Diamond a &\iff \forall \pi \in \text{Paths}(s). \pi \models \Diamond a \\ &\iff \forall \pi \in \text{Paths}(s). \exists j \geq 0. \pi[j] \models a \\ &\implies \forall \pi \in \text{Paths}(s). \exists j \geq 0. (\pi[j] \models a \text{ or } \pi[j] \models b) \\ &\iff \forall \pi \in \text{Paths}(s). \exists j \geq 0. (\pi[j] \models (a \vee b)) \\ &\iff \forall \pi \in \text{Paths}(s). \pi \models \Diamond(a \vee b) \\ &\iff s \models \forall \Diamond(a \vee b) \end{aligned}$$

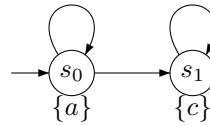
Case 2 is fully analogous (replace a by b).

- However, the other direction, i.e. $\Phi_2 \implies \Phi_1$ does not hold. Consider the following transition system:



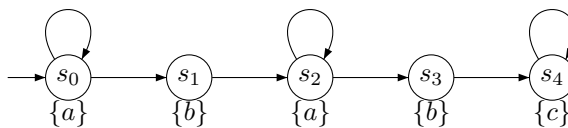
Then, $TS \models \Phi_2$ is fulfilled, but $TS \not\models \Phi_1$.

- (b) – We have $\Psi_1 \not\equiv \Psi_2$: The following transition system provides a counterexample:



Obviously, $TS \models \Psi_1$. Because the proposition b does not hold in any state of TS , c has to hold in an initial state. As this is not the case, the implication is not true (i.e. $TS \not\models \Psi_2$).

- $\Psi_2 \not\equiv \Psi_1$. Again by counterexample:



Here, $TS \models \Psi_2$, but $TS \not\models \Psi_1$:

$Traces(TS) = \mathcal{L}_\omega(\{a\}^+\{b\}\{a\}^+\{b\}\{c\}^\omega)$, but once we enter state s_1 , a does not hold, but $s_1 \not\models \exists(bUc)$. Therefore $TS \not\models \Psi_1$.

Exercise 4 (CTL Normal Forms):

(2 points)

Transform the CTL-formula $\Phi = \neg\forall\Diamond(\forall(\forall\Box b)U(\forall\Diamond a))$ into an equivalent CTL-formula in

- (a) existential normal form and
- (b) positive normal form.

Solution: _____

- (a) Transformation into an equivalent CTL-formula in existential normal form:

$$\begin{aligned} \neg\forall\Diamond(\forall(\forall\Box b)U(\forall\Diamond a)) &\equiv \exists\Box\neg[\forall(\forall\Box b)U(\forall\Diamond a)] \\ &\equiv \exists\Box\neg[\neg\exists((\neg\forall\Diamond a)U(\neg\forall\Box b \wedge \neg\forall\Diamond a)) \wedge \neg\exists\Box\neg\forall\Diamond a] \\ &\equiv \exists\Box\neg[\neg\exists((\exists\Diamond\neg a)U(\exists\Diamond\neg b \wedge \exists\Diamond\neg a)) \wedge \neg\exists\Box\exists\Diamond\neg a] \\ &\equiv \exists\Box\neg[\neg\exists((\exists\Diamond\neg a)U(\exists(trueU\neg b))) \wedge \exists\Diamond\neg a \wedge \neg\exists\Box\exists\Diamond\neg a] \end{aligned}$$

- (b) In positive normal form, we can express Φ as follows:

$$\begin{aligned} \neg\forall\Diamond(\forall(\forall\Box b)U(\forall\Diamond a)) &\equiv \neg\forall[trueU[\forall(\forall\Box b)U(\forall\Diamond a)]] \\ &\equiv \exists[\neg[\forall(\forall\Box b)U(\forall\Diamond a)]Wfalse] \\ &\equiv \exists[\exists[\neg[\forall\Box b \wedge \neg\forall\Diamond a]W(\neg\forall\Box b \wedge \neg\forall\Diamond a)]Wfalse] \\ &\equiv \exists[\exists[(\forall(bWfalse) \wedge \exists\Diamond\neg a)W(\exists\Diamond\neg b \wedge \exists\Diamond\neg a)]Wfalse] \\ &\equiv \exists[\exists[(\forall(bWfalse) \wedge \exists\Diamond\neg a)W(\exists trueU\neg b \wedge \exists\Diamond\neg a)]Wfalse] \end{aligned}$$