

Exercise 1 Semantics of HML with recursion

1) Structural Induction. To show: $S_1 \subseteq S_2 \Rightarrow \llbracket F \rrbracket(S_1) \subseteq \llbracket F \rrbracket(S_2)$.

Induction Base:

$$\begin{aligned}\llbracket X \rrbracket(S_1) &= S_1 \subseteq S_2 = \llbracket X \rrbracket \\ \llbracket \text{tt} \rrbracket(S_1) &= S \subseteq S = \llbracket \text{tt} \rrbracket \\ \llbracket \text{ff} \rrbracket(S_1) &= \emptyset \subseteq \emptyset = \llbracket \text{ff} \rrbracket\end{aligned}$$

Inductive Hypothesis (I.H): $S_1 \subseteq S_2 \Rightarrow \llbracket F_1 \rrbracket(S_1) \subseteq \llbracket F_1 \rrbracket(S_2)$ and $\llbracket F_2 \rrbracket(S_1) \subseteq \llbracket F_2 \rrbracket(S_2)$

Inductive Step:

$$\begin{aligned}\llbracket F_1 \wedge F_2 \rrbracket(S_1) &\stackrel{\text{Def.}}{=} \llbracket F_1 \rrbracket(S_1) \cap \llbracket F_2 \rrbracket(S_1) \\ &\stackrel{\text{I.H.}}{\subseteq} \llbracket F_1 \rrbracket(S_2) \cap \llbracket F_2 \rrbracket(S_2) \\ &\stackrel{\text{Def.}}{=} \llbracket F_1 \wedge F_2 \rrbracket(S_2) \\ \llbracket F_1 \vee F_2 \rrbracket &: \text{analogous} \\ \llbracket \langle a \rangle F_1 \rrbracket(S_1) &\stackrel{\text{Def.}}{=} \langle \cdot a \cdot \rangle(\llbracket F_1 \rrbracket(S_1)) \\ &\stackrel{\text{Def.}}{=} \{s \in S \mid \exists s' \in \llbracket F_1 \rrbracket(S_1) : s \xrightarrow{a} s'\} \\ &\stackrel{*}{\subseteq} \{s \in S \mid \exists s' \in \llbracket F_1 \rrbracket(S_2) : s \xrightarrow{a} s'\} \\ &\stackrel{\text{Def.}}{=} \langle \cdot a \cdot \rangle(\llbracket F_1 \rrbracket(S_2)) \stackrel{\text{Def.}}{=} \llbracket \langle a \rangle F_1 \rrbracket(S_2) \\ \llbracket [a] F_1 \rrbracket &: \text{analogous}\end{aligned}$$

*By I.H.: $S_1 \subseteq S_2 \Rightarrow (s' \in \llbracket F_1 \rrbracket(S_1) \Rightarrow s' \in \llbracket F_1 \rrbracket(S_2))$

□

Complement:

$$\begin{aligned}\llbracket \neg F \rrbracket(T) &:= S \setminus (\llbracket F \rrbracket(T)) \\ \llbracket \neg F \rrbracket(S_1) &:= S \setminus (\llbracket F \rrbracket(S_1)) \\ \llbracket \neg F \rrbracket(S_2) &:= S \setminus (\llbracket F \rrbracket(S_2))\end{aligned}$$

Then, $S_1 \subseteq S_2 \Rightarrow \llbracket F \rrbracket(S_1) \subseteq \llbracket F \rrbracket(S_2)$ by I.H. $\Rightarrow S \setminus (\llbracket F \rrbracket(S_2)) \subseteq S \setminus (\llbracket F \rrbracket(S_1)) \Rightarrow$ Not monotone for $S_1 \neq S_2$.

2) a)

$$\begin{aligned}&\llbracket \langle b \rangle [a] \text{tt} \wedge \langle b \rangle [b] X(\{s_0, s_2\}) \rrbracket \\ &= \langle \cdot b \cdot \rangle(\llbracket [a] \rrbracket(\text{tt})(\{s_0, s_2\})) \cap \langle \cdot b \cdot \rangle(\llbracket [b] \rrbracket(\llbracket X \rrbracket(\{s_0, s_2\}))) \\ &= \langle \cdot b \cdot \rangle(\llbracket [a] \rrbracket(S) \cap \langle \cdot b \cdot \rangle(\llbracket [b] \rrbracket(\{s_0, s_2\}))) \\ &= \langle \cdot b \cdot \rangle(S) \cap \langle \cdot b \cdot \rangle(S) \\ &= \{s_1, s_2\}\end{aligned}$$

b)

$$\begin{aligned}F^0(\emptyset) &= \emptyset \\ F^1(\emptyset) &= \llbracket \langle b \rangle \langle a \rangle \text{tt} \vee \langle b \rangle [b] X \rrbracket(\emptyset) \\ &= \langle \cdot b \cdot \rangle(\langle \cdot a \cdot \rangle(S)) \cup \langle \cdot b \cdot \rangle(\llbracket [b] \rrbracket(\emptyset)) \\ &= \langle \cdot b \cdot \rangle(\{s_0, s_1, s_3\}) \cup \langle \cdot b \cdot \rangle(\{s_0, s_3\}) \\ &= \{s_1, s_2\} \\ F^2(\emptyset) &= \dots = \langle \cdot b \cdot \rangle(\{s_0, s_1, s_3\}) \cup \langle \cdot b \cdot \rangle(\{s_0, s_1, s_3\}) \\ &= \{s_1, s_2\} = F^1(\emptyset)\end{aligned}$$

The fixpoint is $\{s_1, s_2\}$.

c)

	$\llbracket [a] \rrbracket(T_2)$	$\langle \cdot a \cdot \rangle(T_3) \cap \llbracket [b] \rrbracket(T_2)$	$\llbracket [b] \rrbracket(T_2)$
0	S	S	S
1	S	$\{s_0, s_1, s_3\} \cap S$	S
2	$\{s_0, s_1, s_2\}$	$\{s_0, s_1, s_3\} \cap \{s_0, s_3\}$	$\{s_0, s_3\}$
3	$\{s_1, s_2\}$	$\{s_0, s_1, s_3\} \cap \{s_0, s_3\}$	$\{s_0, s_3\}$
4	$\{s_1, s_2\}$	$\{s_0, s_3\}$	$\{s_0, s_3\}$

Exercise 2 Complete Lattices

" $\Leftarrow$ ": Assume for each $Y \subseteq M$: (*) $\gamma(\cap Y) = \cap \{\gamma(l) \mid l \in Y\}$.

i) Let $d_1 \sqsubseteq d_2; d_1, d_2 \in M$. Then $\gamma(d_1) = \gamma(\cap \{d_1, d_2\}) \stackrel{*}{=} \cap \{\gamma(d_1), \gamma(d_2)\} \sqsubseteq \gamma(d_2)$.

ii) (Note: $\cap \emptyset = \top$, because $\forall l \in \emptyset : \top \sqsubseteq l \wedge \forall l' (\forall l'' \in \emptyset \rightarrow l' \sqsubseteq l'') \rightarrow l' \sqsubseteq \top$ (since $l' \in L \setminus M$))

$$\gamma(\top) = \gamma(\cap \emptyset) \stackrel{*}{=} \cap \{\gamma(l) \mid l \in \emptyset\} = \cap \emptyset = \top$$

iii) Let $m_1, m_2 \in M$. Then $\gamma(\cap \{m_1, m_2\}) \stackrel{*}{=} \cap \{\gamma(m_1), \gamma(m_2)\}$.

" $\Rightarrow$ ": Assume i), ii), iii) hold. Proof by induction on the finite cardinality of $M' \subseteq M$:

$$\gamma(\cap M') = \cap \{\gamma(m) \mid m \in M'\} \quad (**)$$

Induction Base: $|M'| = 0$. Then $M' = \emptyset$ and since $\cap \emptyset = \top$, (**) follows from ii).

Inductive Hypothesis (I.H): For an arbitrary but fixed $n \in \mathbb{N}$ (**) holds if $|M'| = n$.

Inductive Step: Let $M' = M'' \cup \{m''\}, m'' \notin M'', |M''| = n$. Then

$$\begin{aligned}\gamma(\cap M') &\stackrel{\text{Def.}}{=} \gamma(\cap \{\cap M'', m''\}) \\ &\stackrel{\text{iii}}{=} \cap \{\gamma(\cap M''), \gamma(m'')\} \\ &\stackrel{\text{I.H.}}{=} \cap \{\cap \{\gamma(l) \mid l \in M''\}, \gamma(m'')\} \\ &= \cap \{\gamma(l) \mid l \in M'' \cup \{m''\}\} \\ &= \cap \{\gamma(l) \mid l \in M'\}\end{aligned}$$

□

Exercise 3 Tarski's Fixed Point Theorem

Let $\text{Ext}(f) \triangleq \{d \in D \mid d \sqsubseteq f(d)\}$.

1. Show that $\text{FIX}(f) \sqsubseteq f(\text{FIX}(f))$: $\forall d \in \text{Ext}(f) : d \overset{\text{Def. Ext}(f)}{\sqsubseteq} f(d) \overset{\text{LUB of Ext}(f)}{\subseteq} \text{FIX}(f) \overset{f \text{ monotone}}{\sqsubseteq} f(\text{FIX}(f))$.
2. Show that $\text{FIX}(f) \supseteq f(\text{FIX}(f))$: $f(\text{FIX}(f)) \overset{1., f \text{ monotone}}{\sqsubseteq} f(f(\text{FIX}(f))) \Rightarrow f(\text{FIX}(f)) \in \text{Ext}(f) \overset{\text{FIX}(f) \text{ LUB of Ext}(f)}{\Rightarrow} \text{FIX}(f) \supseteq f(\text{FIX}(f))$
3. $\text{FIX}(f)$ is greatest: By definition $\text{FIX}(f) = \sqcup \text{Ext}(f) = \sqcup \{d \in D \mid d \sqsubseteq f(d)\} \supseteq \sqcup \{d \in D \mid d = f(d)\} \Rightarrow \forall d : f(d) = d \rightarrow d \sqsubseteq \text{FIX}(f)$

□