

Exercise 1 (Sigma Algebras):

(4 points)

Let Ω be a countably infinite set and define $\mathfrak{F}_\Omega$ as the smallest class of subsets of Ω such that for all $A \subseteq \Omega$

- (i) if A is finite, then $A \in \mathfrak{F}_\Omega$, and
 - (ii) if $A \in \mathfrak{F}_\Omega$, then $A^c \in \mathfrak{F}_\Omega$ for $A^c := (\Omega \setminus A)$.
- a) Show that the definition is non-trivial, i.e., in general $\mathfrak{F}_\Omega \neq 2^\Omega$.
(Hint: find a set Ω and a subset $A \subseteq \Omega$ which cannot be in $\mathfrak{F}_\Omega$ according to the above definition.)
- b) Would this change if $\mathfrak{F}_\Omega$ is defined as the *largest* class of subsets defined as above (instead of the *smallest*)?
- c) Prove or disprove that $\mathfrak{F}_\Omega$ is a σ -algebra as defined in the lecture for any countably infinite set Ω .

Solution: _____

- a) Observe that for any element $A \in \mathfrak{F}_\Omega$ either A itself or A^c has to be finite.
Now consider $\Omega = \mathbb{N}$. The set of even numbers and its complement are both infinite sets and therefore both not in $\mathfrak{F}_\Omega$. Hence we have strict inclusion $\mathfrak{F}_\Omega \subsetneq 2^\Omega$.
- b) If $\mathfrak{F}_\Omega$ were allowed to be the largest class of subsets satisfying the conditions, then the definition is trivial in the sense that $\mathfrak{F}_\Omega = 2^\Omega$.
- c) In general, $\mathfrak{F}_\Omega$ is not a σ -algebra. As a counterexample, we again use $\Omega = \mathbb{N}$. Then, the set of even numbers can obviously be constructed by a countable union of finite sets, but is not contained in $\mathfrak{F}_\Omega$.

Exercise 2 (Geometric Distribution):

(3 + 3 points)

Recall the definition of a *geometric distribution* as given in the lecture:

Definition 1. Let X be a discrete random variable, $k \in \mathbb{N}_{>0}$ and $0 < p \leq 1$. The mass function of a geometric distribution is given by:

$$\Pr\{X = k\} = (1 - p)^{k-1} \cdot p$$

Let X now be geometrically distributed with parameter p .

- a) Show that $E[X] = \frac{1}{p}$ and $\text{Var}[X] = \frac{1-p}{p^2}$.
- b) Prove that

$$\Pr\{X = k + m \mid X > m\} = \Pr\{X = k\} \quad \text{for any } m, k \in \mathbb{N}_{>0}.$$

Hint: Use properties of probability measures and the geometric distribution as presented in the lecture.

Solution: _____

a) $E[X] = \sum_{k=1}^{\infty} k \cdot \Pr(X = k)$

$$\begin{aligned} E[X] &= \sum_{k=1}^{\infty} kp(1-p)^{k-1} \\ &= \sum_{k=0}^{\infty} (k+1)p(1-p)^k \\ &= \sum_{k=0}^{\infty} kp(1-p)^k + \sum_{k=1}^{\infty} p(1-p)^{k-1} \\ &= (1-p)E[X] + 1 \\ &\Rightarrow E[X] = \frac{1}{p} \end{aligned}$$

$$\text{Var}[X] = E[X^2] - E[X]^2$$

$$\begin{aligned} E[X^2] &= \sum_{k=1}^{\infty} k^2 \cdot p(1-p)^{k-1} \\ &= \sum_{k=0}^{\infty} (k+1)^2 \cdot p(1-p)^k \\ &= p + \sum_{k=1}^{\infty} (k+1)^2 \cdot p(1-p)^k \\ &= p + \sum_{k=1}^{\infty} (k^2 + 2k + 1) \cdot p(1-p)^k \\ &= p + \sum_{k=1}^{\infty} k^2 \cdot p(1-p)^k + 2 \sum_{k=1}^{\infty} k \cdot p(1-p)^k + \sum_{k=1}^{\infty} p(1-p)^k \\ &= p + (1-p)(E[X^2] + 2E[X] + 1) \\ &\Rightarrow E[X^2] = 1 + 2\frac{1-p}{p^2} + \frac{1-p}{p} \\ &\Rightarrow \text{Var}(X) = E[X^2] - E[X]^2 = \frac{1-p}{p^2} \end{aligned}$$

b) Presented in the lecture.