

Exercise Sheet 4

General remarks:

- **Due date:** January 13th (before the exercise class). You can either hand your solutions in at the beginning of the exercise class or submit them via email to

`prop1617@i2.informatik.rwth-aachen.de`

- Solutions must be written in English.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Modelling via Conditional Programs)

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Consider the following problem:

A box contains three coins: two regular (i.e. fair) coins and one fake two-headed coin (i.e. $\Pr(\text{heads}) = 1$). You pick a coin at random and toss it, and get heads.

What is the probability that it is the two-headed coin?

- [15%] Write a conditional **pGCL** program to solve this problem.
- [15%] State the post-expectation to measure in order to determine the required probability.

Solution:

```
{w := "fake"} [1/3] {w := "fair"};
ite (w = "fake") {o := 1} {o := 0} [1/2] {o := 1};
observe (o = 1)
```

To determine the probability that the picked coin is the fake one, one needs to consider post-expectation $[w = \text{"fake"}]$.

Exercise 2 (Well-definedness of Expected Rewards)

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Let $(\mathcal{M}, \text{rew})$ be a finite Markov reward model, i.e., $\mathcal{M}$ is a Markov chain with set of states S and $\text{rew} : S \rightarrow \mathbb{N}$ is a function assigning a reward to every state of $\mathcal{M}$. Moreover, let $s \in S$ be a state in $\mathcal{M}$ and $T \subseteq S$ be a set of target states that is reached from s with probability 1, i.e., $\Pr(s \models \Diamond T) = 1$.

Show that the infinite series

$$\text{ExpRew}(s \models \Diamond T) = \sum_{k=0}^{\infty} k \cdot \Pr_s\{\pi \in \text{Paths}(s) \mid \text{rew}(\pi, \Diamond T) = k\}$$

converges.

Here, Paths denotes the set of all paths starting in s . Further, $\text{rew}(\pi, \Diamond T)$ is the accumulated reward along the path π until a state from T is reached. If no such state is reached, we set $\text{rew}(\pi, \Diamond T) = 0$.

Solution: See board.

Exercise 3 (From pGCL with Conditioning to Markov Reward Models)

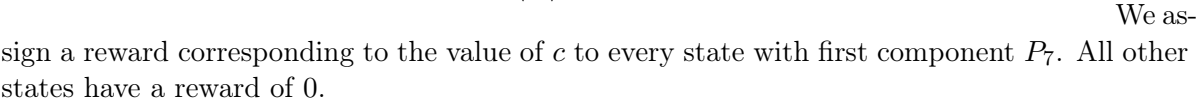
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Consider the following probabilistic program P :

```
 $a, b, c := 0, 0, 0;$   
 $\{c := c + 1; \} [0.5] \{\text{skip}\};$   
 $\{a := 1\} [0.6] \{\text{skip}\};$   
 $\text{if}(a = 1)\{$   
     $\{c := c + 1; b := 1\} [0.2] \{c := c - 1\};$   
 $\}$  else  $\{$   
     $c := 0$   
 $\};$   
 $\text{observe}(a \text{ or } b)$ 
```

- (a) [17.5%] Construct the Markov Reward Model corresponding to P .
- (b) [17.5%] Compute the expected value of c after termination of P .

Solution: We write P_i to denote the sub-program of P that starts with line i . Further, $P_{i,l}$ and $P_{i,r}$ denote the left and right branch of such a program in case of probabilistic choice or conditional statements. To simplify notation, we write 1, 2, 3 instead of $\sigma[a \mapsto 1, b \mapsto 2, c \mapsto 3]$.



Then the expected value of c after eventually reaching $\langle sink \rangle$ under the condition $\neg\langle \frac{1}{2} \rangle$ is

$$\begin{aligned}
 ExpRew(\Diamond\langle sink \rangle \mid \neg\langle \frac{1}{2} \rangle) &= \frac{ExpRew(\Diamond\langle sink \rangle \cap \neg\Diamond\langle \frac{1}{2} \rangle)}{Pr(\neg\Diamond\langle \frac{1}{2} \rangle)} \\
 &= \frac{0.5 \cdot 0.6 \cdot 0.2 \cdot 2 + 0.5 \cdot 0.6 \cdot 0.2 \cdot 1 + 0.5 \cdot 0.6 \cdot 0.8 \cdot (-1)}{1 - (0.5 \cdot 0.4 + 0.5 \cdot 0.4)} \\
 &= \frac{0.12 + 0.06 - 0.24}{0.6} = -0.1
 \end{aligned}$$

Note that this solution is not entirely well-defined, because the post-expectation c is not an expectation according to the strict definition from the lecture.

Exercise 4 (Contextual Equivalence under Conditioning)

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Consider the following pair of programs:

$$\begin{aligned}
 c_1: & \quad x := 1 \\
 c_2: & \quad \{x := 1 \ [1/2] \ x := 0\}; \text{observe } (x=1)
 \end{aligned}$$

They both induce the same probability distribution on variable x (i.e. in both cases x holds value 1 with probability one). Determine whether the following statement is true or false:

For every program context $\mathfrak{C}[\cdot]$, programs $\mathfrak{C}[c_1]$ and $\mathfrak{C}[c_2]$ present always the same behaviour.

Informally, we can understand a *program context* $\mathfrak{C}[\cdot]$ as a program with “holes”, and $\mathfrak{C}[c]$ stands for “filling” the holes with c . In case the above statement is true, prove it. In case it is false, provide a program context $\mathfrak{C}[\cdot]$ and an event (i.e. a postcondition) f that distinguishes $\mathfrak{C}[c_1]$ from $\mathfrak{C}[c_2]$ (i.e. the probability that $\mathfrak{C}[c_1]$ establishes f should be different from that of $\mathfrak{C}[c_2]$).

Solution: The statement is false. If we put c_1 and c_2 into context $\mathfrak{C}[c] = \{c\} \ [1/2] \ \{x := 2\}$, the resulting programs are distinguished e.g. by post-condition $x = 1$, since $\mathfrak{C}[c_1]$ establishes the post-condition with probability $1/2$, while $\mathfrak{C}[c_2]$ with probability $1/3$.