

Exercise Sheet 6

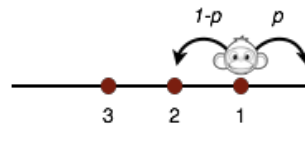
General remarks:

- **Due date:** February 3rd.
- Since there is **no exercise class on January 27th**, you may either submit your solution via email to propro1617@i2.informatik.rwth-aachen.de, or put your solutions into the box labeled *Static Program Analysis* at the chair. In the latter case, please make sure to mark them as submissions to Probabilistic Programming. (before the exercise class).
- Solutions must be written in English.
- If you have any questions regarding the lecture or the exercise, feel free to write us an email or visit us at the chair.

Exercise 1 (Random Walks)

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A monkey is standing one step from the edge of a cliff and takes repeated independent steps; forward, with probability p , or backward, with probability $1-p$, as depicted below:



Show that if $p > 1-p$, then the monkey will almost surely fall off the cliff, i.e. reach position 0.

Solution: Let X_n denote the probability of ever reaching position 0 when starting at position n . By independence we have $X_2 = X_1 \cdot X_1$ since for reaching position 0 from 2, the monkey first needs to reach position 1, which occurs with the same probability X_1 (both cases correspond to a shift of one position to the right), and then move from 1 to 0. Therefore we have

$$X_1 = p \cdot 1 + (1-p) \cdot X_2 = p \cdot 1 + (1-p) \cdot X_1^2.$$

The above quadratic equation on X_1 has two solutions: 1 and $\frac{p}{1-p}$. If $p > 1-p$ the only valid solution is 1 since in this case $\frac{p}{1-p} > 1$.

Exercise 2 (Hardness of lower bound for program expectations)

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Recall that

$$E_{P,\sigma}(f) = \sum_{k=0}^{\infty} \sum_{w \in \{L,R\}^{\leq k}} \wp(T_k(\gamma_{P,\sigma}, w), f)$$

gives the expected value of pGCL program P with respect to post-expectation f on initial state σ . The (outer-most) infinite sum intuitively ranges over the length of all terminating executions.

Given a positive rational number q prove that the problem of deciding whether

$$q < E_{P,\sigma}(f)$$

belongs to Σ_1 .

Solution: For proving the membership $\mathcal{LEX} \in \Sigma_1^0$, consider the following:

$$\begin{aligned} q &< E_{P,\sigma}(f) \\ \iff q &< \sum_{k=0}^{\infty} \sum_{w \in \{L,R\}^{\leq k}} \wp(T_k(\gamma_{P,\sigma}, w), f) \\ \iff \exists y: q &< \underbrace{\sum_{k=0}^y \sum_{w \in \{L,R\}^{\leq k}} \wp(T_k(\gamma_{P,\sigma}, w), f)}_{\text{decidable relation in FOL}} \quad (\text{summands are positive}) \end{aligned}$$

Exercise 3 (Hardness of approximating variances)

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Let P be a pGCL program (with observe), σ be an initial state and f be an expectation. Then the (conditional) *variance* of P with respect to f and σ is defined as

$$\text{Var}[P, \sigma](f) = \underline{\text{cwp}}[P](f^2)(\sigma) - \underline{\text{cwp}}[P](f)(\sigma) \cdot \underline{\text{cwp}}[P](f)(\sigma).$$

Show that computing lower bounds on the variance, i.e. deciding whether $\text{Var}[P, \sigma](f) > q$ for some fixed rational number q , is Σ_2^0 -hard.

Solution: We provide a reduction from $\overline{AST}$ —the complement of almost sure termination. Hence, let (c, σ) be an instance of (the complement of) the almost sure termination problem. Further, consider the reduction function

$$r(c, \sigma) = (c', \sigma, f, q),$$

where $f = v$, v is a fresh variable, $q = 0$ and c' is the program

$$v := 0; \{\text{skip}\} [1/2] \{c\}; v := 1.$$

Then

$$\begin{aligned} \text{Var}[c', \sigma](v) &= \underline{\text{cwp}}[c'](v^2)(\sigma) - \underline{\text{cwp}}[c'](v)(\sigma) \cdot \underline{\text{cwp}}[c'](v)(\sigma) \\ &= \text{wp}[c'](v^2)(\sigma) - \text{wp}[c'](v)(\sigma) \cdot \text{wp}[c'](v)(\sigma) \quad (c' \text{ is observe-free}) \end{aligned}$$

Since v does not occur in c and v is set from 0 to 1 if and only if c' has terminated, this is equal to:

$$\begin{aligned} &= \text{wp}[c'](1^2)(\sigma) - \text{wp}[c'](1)(\sigma) \cdot \text{wp}[c'](1)(\sigma) \\ &= \text{wp}[c'](1)(\sigma) - \text{wp}[c'](1)(\sigma) \cdot \text{wp}[c'](1)(\sigma). \end{aligned}$$

Since $\mathbf{wp}[c'](\mathbf{1})(\sigma)$ is exactly the probability of c' terminating on input σ , we have $\mathit{Var}[c', \sigma](v) > 0$ if and only if c' terminates neither with probability 0 nor with probability 1. By construction, c' terminates with probability at least a half. Thus, $\mathit{Var}[c', \sigma](v) > 0$ if and only if c' does not terminate with probability 1.