

Exercise Sheet 4

Sample Solution

Task 1: Chain Complete Partial Orders

(a): TRUE

Let $d \sqsubseteq_1 d'$. Then $\{d, d'\}$ is a chain.

Thus $f(d') = f(\underbrace{\sqcup_1 \{d, d'\}}_{=d'}) \stackrel{f \text{ continuous}}{=} \sqcup_2 \{f(d), f(d')\} \stackrel{\text{def. L.U.B.}}{\sqsupseteq_2} f(d)$.

Therefore $f(d) \sqsubset_2 f(d')$ holds.

Alternatively: True by Definition 7.15.

(b): FALSE

Let $S = \{x \in \mathbb{Q} \mid x \leq \sqrt{2}\}$.

Then S is a chain, but $\sqcup S = \sqrt{2} \notin \mathbb{Q}$.

(c): FALSE

Let $(D_1, \sqsubseteq) = (\mathbb{N} \cup \{\infty\}, \leq)$ and

$$f : D_1 \rightarrow D_1, x \mapsto \begin{cases} 0, & x < \infty \\ \infty & x = \infty \end{cases}.$$

f is monotonic, because $x \leq y \Rightarrow f(x) = \begin{cases} 0 & \leq f(y) \\ \infty & \leq f(y) \end{cases} \quad \begin{matrix} \text{if } x < \infty \\ \text{if } x = y = \infty \end{matrix}$.

However, $\underbrace{f(\sqcup \mathbb{N})}_{=\infty} \not\sqsubseteq \underbrace{\sqcup f(\mathbb{N})}_{=0}$.

(d): TRUE

Since $f(p) \sqsubseteq p$, it suffices to prove $p \sqsubseteq f(p)$.

First note that $f(p)$ implies $f(f(p)) \sqsubseteq f(p)$

Since p is the least element with $f(p) \sqsubseteq p$, we have $p \sqsubseteq p' = f(p)$.

Thus $p = f(p)$ holds.

$(f(p) = p')$

Task 2: repeat-until Loops

(a)

$$\begin{aligned}
 & \mathfrak{C}[\text{repeat } c \text{ until } b] \\
 &= \mathfrak{C}[c; \text{ if } b \text{ then skip else repeat } c \text{ until } b] \\
 &= \mathfrak{C}[\text{if } b \text{ then skip else repeat } c \text{ until } b] \circ \mathfrak{C}[c] \\
 &= \text{cond}(\mathfrak{B}[b], \mathfrak{C}[\text{skip}], \mathfrak{C}[\text{repeat } c \text{ until } b]) \circ \mathfrak{C}[c] \\
 &= \text{cond}(\mathfrak{B}[b], id_{\Sigma}, \mathfrak{C}[\text{repeat } c \text{ until } b]) \circ \mathfrak{C}[c]
 \end{aligned}$$

Then $\mathfrak{C}[\text{repeat } c \text{ until } b]$ is the least fixed point of F given by:

$$F(f) = \text{cond}(\mathfrak{B}[b], id_{\Sigma}, \mathfrak{C}[\text{repeat } c \text{ until } b]) \circ \mathfrak{C}[c]$$

(b)

$$\begin{aligned}
 \hat{F}(f) &= \text{cond}(\mathfrak{B}[\text{false}], id_{\Sigma}, f) \circ \mathfrak{C}[\text{skip}] \\
 &= f \circ \mathfrak{C}[\text{skip}] \\
 &= f \circ id_{\Sigma} \\
 &= f
 \end{aligned}$$

$\hat{F}$ is the identity transformer.

(c)

Since $\hat{F}(f) = f$ for all $f : \Sigma \dashrightarrow \Sigma$ and $f_{\emptyset} \sqsubseteq f$ for all $f : \Sigma \dashrightarrow \Sigma$, we have $\text{fix}(\hat{F}) = f_{\emptyset}$

Task 3: Closed Sets

(a)

Apply Tarski-Knaster Theorem:

$$\text{fix}(f) = \sqcup \{f^n(\sqcup \emptyset) \mid n \geq 0\}$$

Since $\emptyset$ is a chain and $\emptyset \subseteq C$, we have $\sqcup \emptyset \in C$.

By definition, $f(\sqcup \emptyset) \in C$.

By complete induction, $\forall n. f^n(\sqcup \emptyset) \in C$.

Hence, $G = \{f^n(\sqcup \emptyset) \mid n \geq 0\} \subseteq C$ is a chain.

By definition, $\text{fix}(f) = \sqcup G \in C$.

(b)

Let $C := \{y \in D \mid y \sqsubseteq x\}$. C is closed.

If $y \in D$ then $f(y) \sqsubseteq f(x) \sqsubseteq x$.

Thus $f(y) \in C$. By (a), $\text{fix}(f) \in C$.

By definition of C , $\text{fix}(f) \sqsubseteq x$.