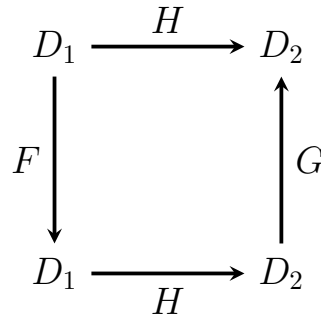


Exercise Sheet 5

Sample Solution

Task 1: Fusion Lemma

(a)



① We first prove that $G^n(\perp_2) = H \circ F^n(\perp_1)$ by induction over n .

Induction Base: $n = 0$

$$G^0(\perp_2) = \perp_2 \stackrel{H \text{ strict}}{=} H(\perp_1) = H(F^0(\perp_1))$$

Induction Hypothesis:

$G^n(\perp_2) = H(F^n(\perp_1))$ holds for any arbitrary but fixed $n \in \mathbb{N}$.

Induction Step: $n \mapsto n + 1$

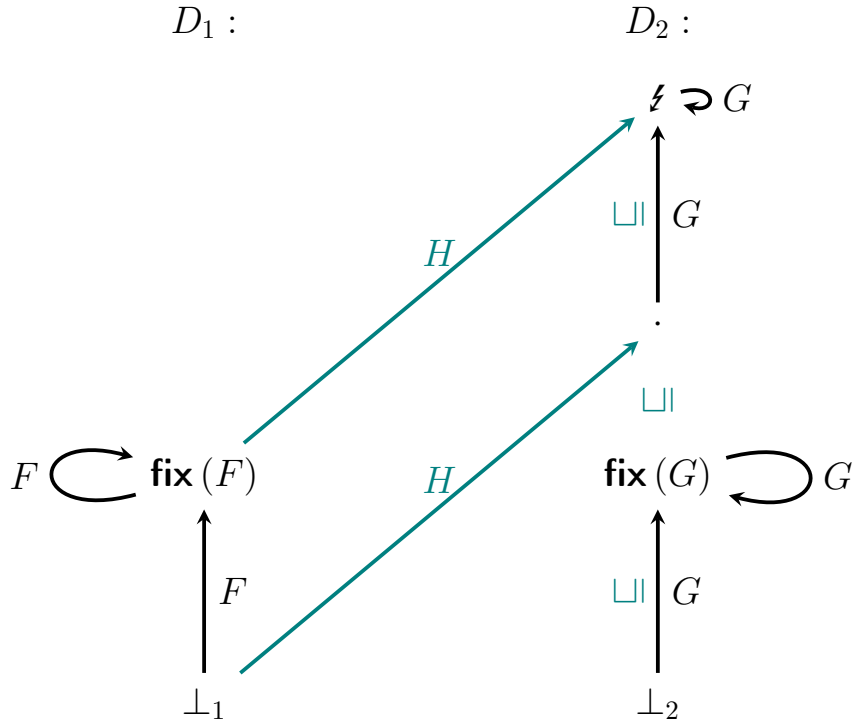
$$\begin{aligned}
 G^{n+1}(\perp_2) &= G \circ G^n(\perp_2) \\
 &\stackrel{I.H.}{=} G \circ H \circ F^n(\perp_1) \\
 &\stackrel{G \circ H = H \circ F}{=} H \circ F \circ F^n(\perp_1) \\
 &= H(F^{n+1}(\perp_1))
 \end{aligned}$$

② Now we prove that $\text{fix}(G) = H(\text{fix}(F))$.

$$\begin{aligned}
 \text{fix}(G) &= \sqcup \{G^n(\perp_2) \mid n \in \mathbb{N}\} && | \text{ by Tarski-Knaster} \\
 &= \sqcup \{H \circ F^n(\perp_1) \mid n \in \mathbb{N}\} && | \text{ by } \textcircled{1} \\
 &= H(\sqcup \{F^n(\perp_1) \mid n \in \mathbb{N}\}) && | \text{ by } H \text{ continuous} \\
 &= H(\text{fix}(F)) && | \text{ by Tarski-Knaster}
 \end{aligned}$$

(b)

Wrong!



G, F, H continuous ✓

$H \circ F = G \circ H$ ✓

$H(\text{fix}(F)) = \text{⚡} \neq \text{fix}(G)$

Task 2: Tarski-Kantorovich Principle

Prove or disprove: Let $(D, \sqsubseteq)$ be a CCPO and let $F : D \rightarrow D$ be continuous. Moreover, let $d \in D$, such that $d \sqsubseteq F(d)$.

Then F has at least one fixpoint larger than d and the least of those fixpoints is given by

$$\sqcup \{F^n(d) \mid n \in \mathbb{N}\}$$

We disprove this assumption by specifying a counterexample.

We define F in such a way that $d = F(d)$ but $d \neq \text{fix}(F)$.

Utilising Tarski-Knaster and the definition of this partial order we get $d \sqsubseteq F(d)$ and $\sqcup \{F^n(d) \mid n \in \mathbb{N}\} = d \not\sqsupseteq d$. ⚡

Prove or disprove: Let $(D, \sqsubseteq)$ be a CCPO and let $F : D \rightarrow D$ be continuous. Moreover, let $d \in D$, such that $d \sqsubseteq F(d)$. Then F has at least one fixpoint larger or equal than d and the least of those fixpoints is given by

$$\sqcup \{F^n(d) \mid n \in \mathbb{N}\}$$

Let $\hat{d} := \sqcup \{F^n(d) \mid n \in \mathbb{N}\}$.

$\hat{d}$ exists, since $d \sqsubseteq F(d) \sqsubseteq F^2(d) \sqsubseteq \dots$ is a chain and D is a CCPO. $\hat{d}$ is a fixpoint since

$$\begin{aligned} F(\hat{d}) &= F(\sqcup \{F^n(d) \mid n \in \mathbb{N}\}) \\ &= \sqcup \{F^{n+1}(d) \mid n \in \mathbb{N}\} && | \text{ by continuity of } F \\ &= \sqcup \{F^n(d) \mid n \in \mathbb{N}\} && | d \sqsubseteq F(d) \\ &= \hat{d} \end{aligned}$$

$\hat{d}$ is least fixpoint “above” d

Let x be another fixpoint “above” d “smaller” than $\hat{d}$, i.e.

- (1) $f(x) = x$
- (2) $d \sqsubseteq x$
- (3) $x \sqsubseteq \hat{d}$

But:

$$\begin{aligned} d \sqsubseteq x &\Rightarrow F(d) \sqsubseteq F(x) && | F \text{ monoton} \\ &\Rightarrow \forall n \in \mathbb{N} (F^n(d) \sqsubseteq F^n(x)) && | \\ &\Rightarrow \sqcup \{F^n(d) \mid n \in \mathbb{N}\} \sqsubseteq \sqcup \{F^n(x) \mid n \in \mathbb{N}\} \\ &\Rightarrow \hat{d} \sqsubseteq \sqcup \{F^n(x) \mid n \in \mathbb{N}\} \\ &\Rightarrow \hat{d} \sqsubseteq \sqcup \{x \mid n \in \mathbb{N}\} = \{x\} && | \text{ by (1)} \\ &\Rightarrow \hat{d} \sqsubseteq x \\ &\Rightarrow \hat{d} = x && | \text{ by (3) and} \\ &&& \text{anti-symmetry of} \\ &&& \text{CCPO} \end{aligned}$$

Task 3: Complete Lattice

Every chain is a subset

Thus: Complete Lattices are chain complete partial orders (**CCPO**)

$\Rightarrow$ Tarski-Knaster is applicable.