



Foundations of Informatics: a Bridging Course

Week 3: Formal Languages and Processes

Part C: Context-Free Languages

March 31 – April 4, 2025

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<https://moves.rwth-aachen.de/teaching/ws-2024-25/foi/>

Outline of Part C

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

Chomsky Normal Form

The Word Problem for Context-Free Languages

The Emptiness Problem for Context-Free Languages

Closure Properties of Context-Free Languages

Pushdown Automata

Pushdown Automata and Context-Free Languages

Introductory Example I

Example C.1

Syntax definition of programming languages by “Backus-Naur” rules

Here: **simple arithmetic expressions**

$$\begin{array}{lcl} \langle \textit{Expression} \rangle & ::= & 0 \\ & | & 1 \\ & | & \langle \textit{Expression} \rangle + \langle \textit{Expression} \rangle \\ & | & \langle \textit{Expression} \rangle * \langle \textit{Expression} \rangle \\ & | & (\langle \textit{Expression} \rangle) \end{array}$$

Meaning:

An expression is either 0 or 1, or it is of the form $u + v$, $u * v$, or (u) where u, v are again expressions

Introductory Example II

Example C.1 (continued)

Here we abbreviate $\langle \textit{Expression} \rangle$ as E , and use “ $\rightarrow$ ” instead of “ $::=$ ”. Thus:

$$E \rightarrow 0 \mid 1 \mid E + E \mid E * E \mid (E)$$

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Example C.1 (continued)

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Now expressions can be generated by replacing nonterminal symbols according to rules, beginning with the start symbol E :

$$E \Rightarrow E * E$$

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Context-Free Grammars I

Definition C.2

A **context-free grammar (CFG)** is a quadruple

$$G = \langle N, \Sigma, P, S \rangle$$

where

- N is a finite set of **nonterminal symbols**
- Σ is the (finite) alphabet of **terminal symbols** (disjoint from N)
- P is a finite set of **production rules** of the form $A \rightarrow \alpha$ where $A \in N$ and $\alpha \in (N \cup \Sigma)^*$
- $S \in N$ is a **start symbol**

Example C.3

For the above example, we have:

- $N = \{E\}$
- $\Sigma = \{0, 1, +, *, (,)\}$
- $P = \{E \rightarrow 0, E \rightarrow 1, E \rightarrow E + E, E \rightarrow E * E, E \rightarrow (E)\}$
- $S = E$

Context-Free Grammars II

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Naming conventions:

- nonterminals start with uppercase letters
 - terminals start with lowercase letters
 - start symbol = symbol on LHS of first production
- ⇒ grammar completely defined by productions

Context-Free Languages I

Definition C.4

Let $G = \langle N, \Sigma, P, S \rangle$ be a CFG.

- A **sentence** $\gamma \in (N \cup \Sigma)^*$ is **directly derivable** from $\beta \in (N \cup \Sigma)^*$ if there exist $\pi = A \rightarrow \alpha \in P$ and $\delta_1, \delta_2 \in (N \cup \Sigma)^*$ such that $\beta = \delta_1 A \delta_2$ and $\gamma = \delta_1 \alpha \delta_2$ (notation: $\beta \xRightarrow{\pi} \gamma$ or just $\beta \Rightarrow \gamma$).

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- A **derivation** (of length $n \in \mathbb{N}$) of γ from β is a sequence of direct derivations of the form $\delta_0 \Rightarrow \delta_1 \Rightarrow \dots \Rightarrow \delta_n$ where $\delta_0 = \beta$, $\delta_n = \gamma$, and $\delta_{i-1} \Rightarrow \delta_i$ for every $i \in \{1, \dots, n\}$ (notation: $\beta \Rightarrow^* \gamma$).

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- A word $w \in \Sigma^*$ is called **derivable** in G if $S \Rightarrow^* w$.

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- A word $w \in \Sigma^*$ is called **derivable** in G if $S \Rightarrow^* w$.
- The **language generated by** G is $L(G) := \{w \in \Sigma^* \mid S \Rightarrow^* w\}$.
- A language $L \subseteq \Sigma^*$ is called **context-free (CFL)** if it is generated by some CFG.
- Two grammars G_1, G_2 are **equivalent** if $L(G_1) = L(G_2)$.

Context-Free Languages II

Example C.5

The language

$$\{a^n b^n \mid n \in \mathbb{N}\}$$

is context-free. It is generated by the grammar $G = \langle N, \Sigma, P, S \rangle$ with

- $N = \{S\}$
- $\Sigma = \{a, b\}$
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(proof: generating $a^n b^n$ requires exactly n applications of the first and one concluding application of the second rule)

Context-Free Languages II

Example C.5

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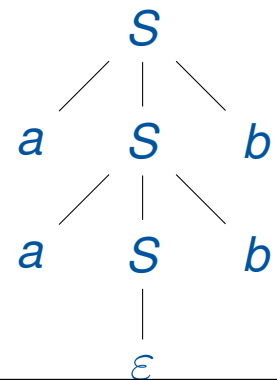
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Remark: illustration of derivations by **derivation trees**

- root labelled by start symbol
- leaves labelled by terminal symbols
- successors of node labelled according to right-hand side of production rule
- sequence of leaf symbols = generated word



Summary: Context-Free Grammars and Languages

Seen:

- Context-free grammars
- Derivations
- Context-free languages

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Next:

- Relation between context-free and regular languages

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Theorem C.6

1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a **proper subset** of CFLs.)

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1. *Every regular language is context-free.*
2. *There exist CFLs which are not regular.*

(Thus: regular languages are a **proper subset** of CFLs.)

Proof.

1. Let L be a regular language, and let $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$ be a DFA which recognises L . $G_{\mathcal{A}} := \langle N, \Sigma, P, S \rangle$ is defined as follows:
 - $N := Q, S := q_0$
 - if $\delta(q, a) = q'$, then $q \rightarrow aq' \in P$
 - if $q \in F$, then $q \rightarrow \varepsilon \in P$

Obviously a w -labelled run in $\mathcal{A}$ from q_0 to F corresponds to a derivation of w in $G_{\mathcal{A}}$, and vice versa. Thus $L(\mathcal{A}) = L(G_{\mathcal{A}})$ (example on the following slide).

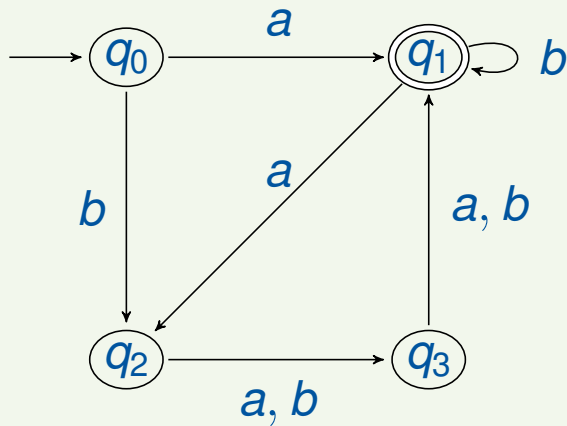
2. An example is $\{a^n b^n \mid n \in \mathbb{N}\}$ (see Lesson 1).

Intuitive reason for non-regularity: recognising this language requires “unbounded counting” capability. □

From Regular to Context-Free Languages

Example C.7

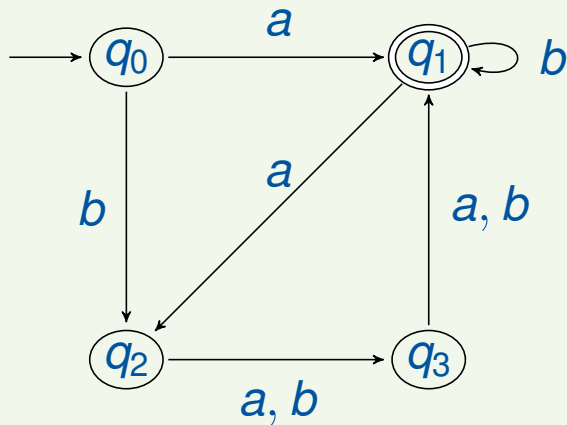
DFA $\mathcal{A} = \langle Q, \Sigma, \delta, q_0, F \rangle$:



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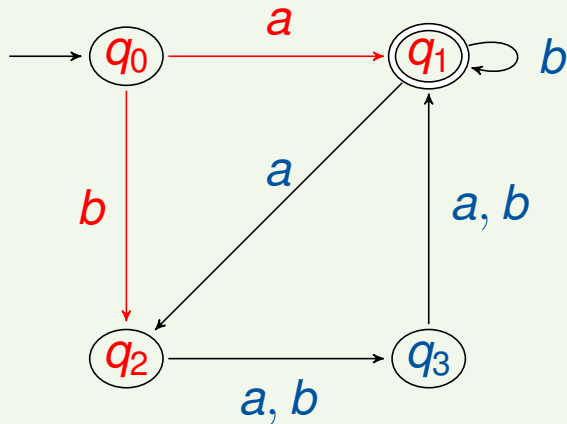


Corresponding CFG $G_{\mathcal{A}} := \langle N, \Sigma, P, S \rangle$
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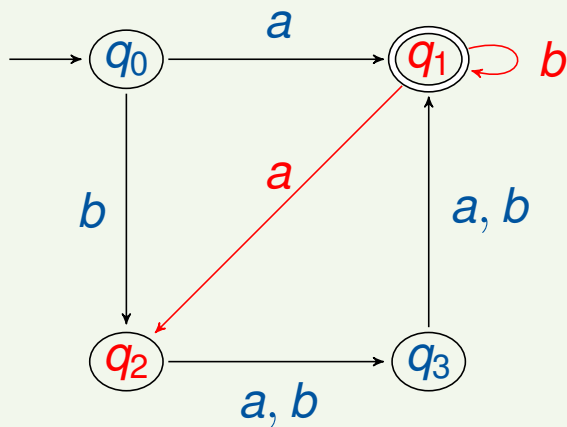
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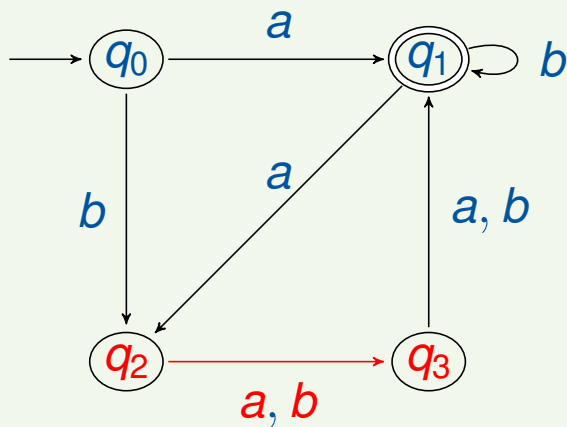
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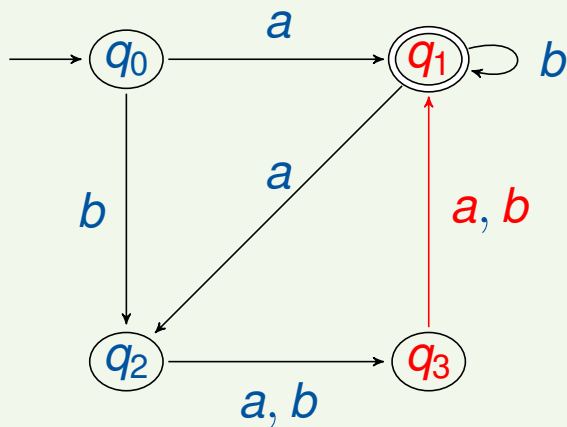
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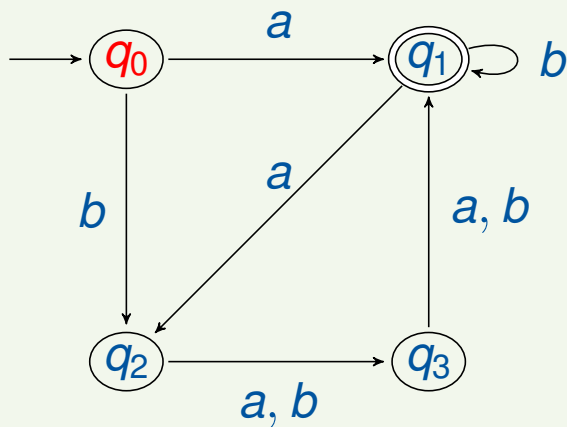
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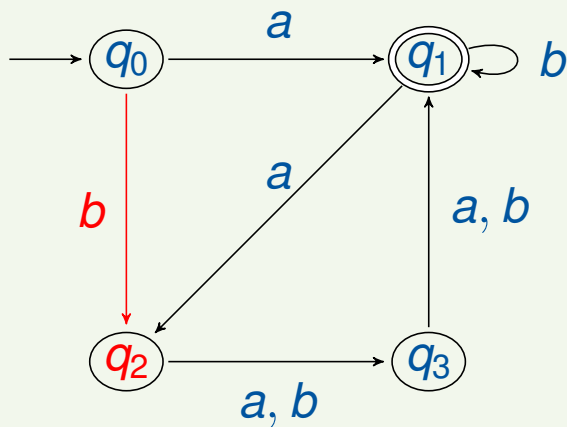
E.g., $\mathcal{A}$'s run on input $baab \in L(\mathcal{A})$ is simulated by the following derivation in $G_{\mathcal{A}}$:

q_0

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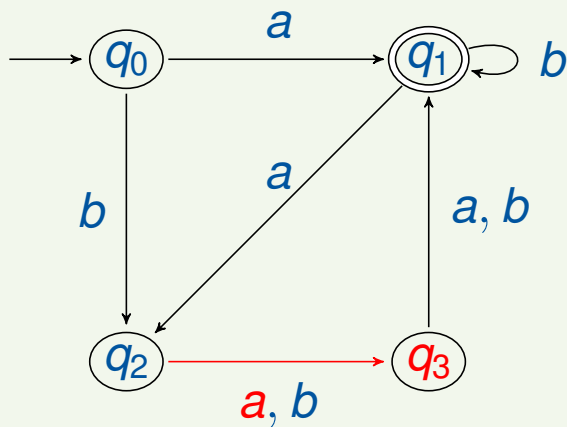
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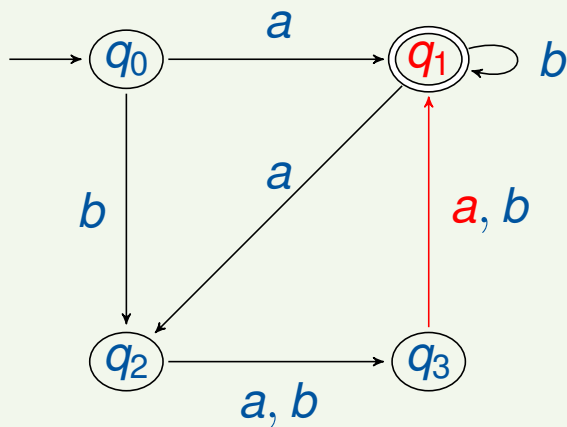
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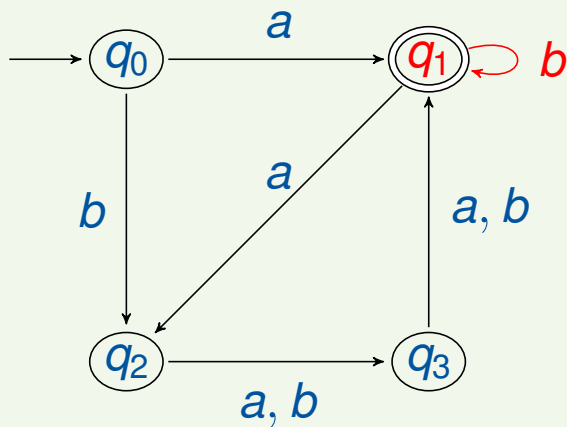
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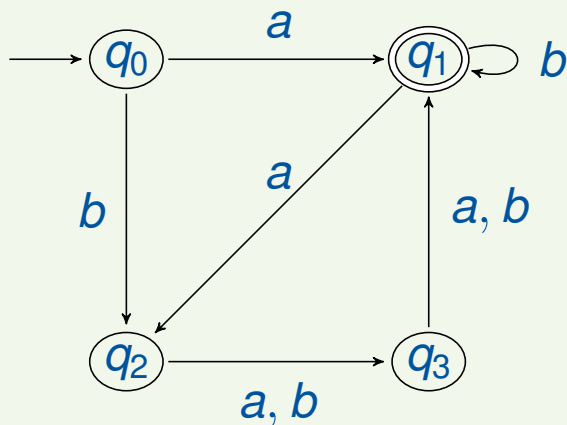
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- CFLs are more expressive than regular languages

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Next:

- Decidability of word problem

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- Important problem with many applications
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 - ...
- For regular languages this was easy: just let the corresponding DFA run on w .
- But here: how to decide **when to stop** a derivation?

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Given CFG $G = \langle N, \Sigma, P, S \rangle$ and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not.

- Important problem with many applications
 - syntax analysis of programming languages
 - HTML parsers
 - ...
 - For regular languages this was easy: just let the corresponding DFA run on w .
 - But here: how to decide **when to stop** a derivation?
 - **Solution:** establish **normal form** for grammars which guarantees that each nonterminal produces at least one terminal symbol
- ⇒ Only **finitely many combinations** to be inspected

Chomsky Normal Form

Definition C.8

A CFG is in **Chomsky Normal Form (Chomsky NF)** if every of its productions is of the form

$$A \rightarrow BC \quad \text{or} \quad A \rightarrow a$$

Chomsky Normal Form

Definition C.8

A CFG is in **Chomsky Normal Form (Chomsky NF)** if every of its productions is of the form

$$A \rightarrow BC \quad \text{or} \quad A \rightarrow a$$

Example C.9

Consider the grammar $S \rightarrow ab \mid aSb$, which generates $L := \{a^n b^n \mid n \geq 1\}$.
An equivalent grammar in Chomsky NF is

$$\begin{array}{ll} S \rightarrow AB \mid AC & (\text{generates } L) \\ A \rightarrow a & (\text{generates } \{a\}) \\ B \rightarrow b & (\text{generates } \{b\}) \\ C \rightarrow SB & (\text{generates } \{a^n b^{n+1} \mid n \geq 1\}) \end{array}$$

Conversion to Chomsky Normal Form

Theorem C.10

Every CFL L (without ε -productions) can be generated by a CFG in Chomsky NF.

Conversion to Chomsky Normal Form

Theorem C.10

Every CFL L (without ε -productions) can be generated by a CFG in Chomsky NF.

Proof.

Let L be a CFL, and let $G = \langle N, \Sigma, P, S \rangle$ be some CFG which generates L . The transformation of P into rules of the form $A \rightarrow BC$ and $A \rightarrow a$ proceeds in three steps:

1. terminal symbols only in rules of the form $A \rightarrow a$
(thus all other rules have the shape $A \rightarrow A_1 \dots A_n$)
2. elimination of “chain rules” of the form $A \rightarrow B$
3. elimination of rules of the form $A \rightarrow A_1 \dots A_n$ where $n > 2$

(see following slides for details)



Step 1: Only $A \rightarrow a$

Procedure

1. For every terminal symbol $a \in \Sigma$, introduce a new nonterminal symbol $B_a \in N$.
2. Add corresponding productions $B_a \rightarrow a$ to P .
3. In each original production $A \rightarrow \alpha$, replace every $a \in \Sigma$ with B_a .

This yields G' .

Step 1: Only $A \rightarrow a$

Procedure

1. For every terminal symbol $a \in \Sigma$, introduce a new nonterminal symbol $B_a \in N$.
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3. In each original production $A \rightarrow \alpha$, replace every $a \in \Sigma$ with B_a .

This yields G' .

Example C.11

$G : S \rightarrow ab \mid aSb$ is converted to $G' : S \rightarrow AB \mid ASB$
 $A \rightarrow a$
 $B \rightarrow b$

Step 2: Elimination of Chain Rules $A \rightarrow B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \dots \Rightarrow A_n$ with rules of the form $A \rightarrow B$ without repetition of nonterminals ($\Rightarrow$ only finitely many!).
2. Determine all productions $A_n \rightarrow \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \rightarrow \alpha$ to P .
4. Remove all chain rules from P .

This yields G' .

Step 2: Elimination of Chain Rules $A \rightarrow B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \dots \Rightarrow A_n$ with rules of the form $A \rightarrow B$ without repetition of nonterminals ($\Rightarrow$ only finitely many!).
2. Determine all productions $A_n \rightarrow \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \rightarrow \alpha$ to P .
4. Remove all chain rules from P .

This yields G' .

Example C.12

$$\begin{array}{l} G' : S \rightarrow A \\ A \rightarrow B \mid C \\ B \rightarrow A \mid DA \\ C \rightarrow c \\ D \rightarrow d \end{array}$$

Step 2: Elimination of Chain Rules $A \rightarrow B$

Procedure

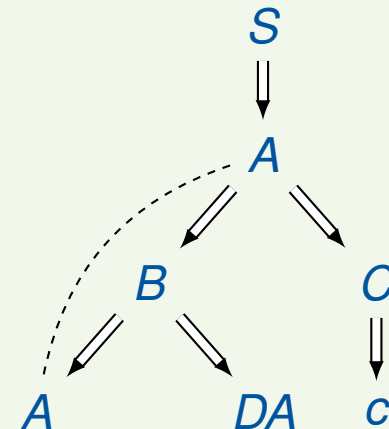
1. Determine all derivations $A_1 \Rightarrow \dots \Rightarrow A_n$ with rules of the form $A \rightarrow B$ without repetition of nonterminals ($\Rightarrow$ only finitely many!).
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3. Add corresponding productions $A_1 \rightarrow \alpha$ to P .
4. Remove all chain rules from P .

This yields G' .

Example C.12

$G' :$

$S \rightarrow A$	
$A \rightarrow B$	C
$B \rightarrow A$	DA
$C \rightarrow c$	
$D \rightarrow d$	



Step 2: Elimination of Chain Rules $A \rightarrow B$

Procedure

1. Determine all derivations $A_1 \Rightarrow \dots \Rightarrow A_n$ with rules of the form $A \rightarrow B$ without repetition of nonterminals ($\Rightarrow$ only finitely many!).
2. Determine all productions $A_n \rightarrow \alpha$ with $\alpha \notin N$.
3. Add corresponding productions $A_1 \rightarrow \alpha$ to P .
4. Remove all chain rules from P .

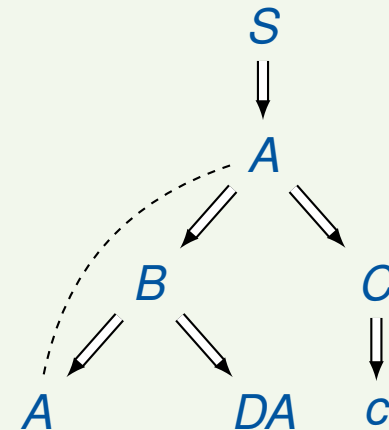
This yields G'' .

Example C.12

is converted to

$$G' : \begin{array}{l} S \rightarrow A \\ A \rightarrow B \mid C \\ B \rightarrow A \mid DA \\ C \rightarrow c \\ D \rightarrow d \end{array}$$

$$G'' : \begin{array}{l} S \rightarrow DA \mid c \\ A \rightarrow DA \mid c \\ B \rightarrow DA \mid c \\ C \rightarrow c \\ D \rightarrow d \end{array}$$



Step 3: Elimination of Rules $A \rightarrow A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \rightarrow A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \rightarrow A_1 B$.
3. Add new production $B \rightarrow A_2 \dots A_n$.

This yields G''' .

Step 3: Elimination of Rules $A \rightarrow A_1 \dots A_n$ with $n > 2$

Procedure

Iteratively apply the following transformation:

1. For every $A \rightarrow A_1 \dots A_n$ with $n > 2$, introduce a new nonterminal symbol $B \in N$.
2. Replace original production by $A \rightarrow A_1 B$.
3. Add new production $B \rightarrow A_2 \dots A_n$.

This yields G''' .

Example C.13

$G'' : S \rightarrow AB \mid ASB$ is converted to $G''' : S \rightarrow AB \mid AC$
 $A \rightarrow a$
 $B \rightarrow b$
 $C \rightarrow SB$

Summary: Chomsky Normal Form

Seen:

- Chomsky NF: all productions of the form $A \rightarrow BC$ or $A \rightarrow a$

Summary: Chomsky Normal Form

Seen:

- Chomsky NF: all productions of the form $A \rightarrow BC$ or $A \rightarrow a$

Next:

- Exploit Chomsky Normal Form to solve word problem for CFL

Outline of Part C

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

Chomsky Normal Form

The Word Problem for Context-Free Languages

The Emptiness Problem for Context-Free Languages

Closure Properties of Context-Free Languages

Pushdown Automata

Pushdown Automata and Context-Free Languages

The Word Problem for CFL

Word Problem for ε -free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

The Word Problem for CFL

Word Problem for ε -free CFL

Given CFG $G = \langle N, \Sigma, P, S \rangle$ such that $\varepsilon \notin L(G)$ and $w \in \Sigma^+$, decide whether $w \in L(G)$ or not.

(If $w = \varepsilon$, then $w \in L(G)$ easily decidable for arbitrary G)

Algorithm C.14 (by Cocke, Younger, Kasami – CYK algorithm)

1. Transform G into Chomsky NF
2. Let $w = a_1 \dots a_n$ ($n \geq 1$)
3. Let $w[i, j] := a_i \dots a_j$ for every $1 \leq i \leq j \leq n$
4. Consider segments $w[i, j]$ in order of increasing length, starting with $w[i, i] = a_i$ (i.e., letters)
5. In each case, determine $N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i, j]\}$ using a “dynamic programming” approach:
 - $i = j$: $N_{i,i} = \{A \in N \mid A \rightarrow a_i \in P\}$
 - $i < j$: $N_{i,j} = \{A \in N \mid \exists B, C \in N, k \in \{i, \dots, j-1\} : A \rightarrow BC \in P, B \in N_{i,k}, C \in N_{k+1,j}\}$
6. Test whether $S \in N_{1,n}$ (and thus, whether $S \Rightarrow^* w[1, n] = w$)

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\dots$	a_n
$i \setminus j$	1	2	3	$\dots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\dots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\dots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\dots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\dots$	$\vdots$
n	X	X	$\dots$	$\dots$	$N_{n,n}$

Matrix Representation of CYK Algorithm

$$\begin{aligned} N_{1,1} &= \{A \in N \mid A \rightarrow a_1 \in P\} \\ N_{2,2} &= \{A \in N \mid A \rightarrow a_2 \in P\} \\ &\vdots \end{aligned}$$

	a_1	a_2	a_3	$\dots$	a_n
$i \backslash j$	1	2	3	$\dots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\dots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\dots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\dots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\dots$	$\vdots$
n	X	X	$\dots$	$\dots$	$N_{n,n}$

Matrix Representation of CYK Algorithm

$$N_{1,1} = \{A \in N \mid A \rightarrow a_1 \in P\}$$

$$N_{2,2} = \{A \in N \mid A \rightarrow a_2 \in P\}$$

⋮

$$N_{1,2} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{1,1}, C \in N_{2,2}\}$$

$$N_{2,3} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{2,2}, C \in N_{3,3}\}$$

⋮

	a_1	a_2	a_3	$\cdots$	a_n
$i \backslash j$	1	2	3	$\cdots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\cdots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\cdots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\cdots$	$N_{3,n}$
⋮	⋮	⋮		$\cdots$	⋮
n	X	X	$\cdots$	$\cdots$	$N_{n,n}$

Matrix Representation of CYK Algorithm

	a_1	a_2	a_3	$\dots$	a_n
$i \backslash j$	1	2	3	$\dots$	n
1	$N_{1,1}$	$N_{1,2}$	$N_{1,3}$	$\dots$	$N_{1,n}$
2	X	$N_{2,2}$	$N_{2,3}$	$\dots$	$N_{2,n}$
3	X	X	$N_{3,3}$	$\dots$	$N_{3,n}$
$\vdots$	$\vdots$	$\vdots$		$\dots$	$\vdots$
n	X	X	$\dots$	$\dots$	$N_{n,n}$

$$N_{1,1} = \{A \in N \mid A \rightarrow a_1 \in P\}$$

$$N_{2,2} = \{A \in N \mid A \rightarrow a_2 \in P\}$$

$\vdots$

$$N_{1,2} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{1,1}, C \in N_{2,2}\}$$

$$N_{2,3} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{2,2}, C \in N_{3,3}\}$$

$\vdots$

$$N_{1,3} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{1,1}, C \in N_{2,3}\}$$

$$\cup \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{1,2}, C \in N_{3,3}\}$$

$$N_{2,4} = \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{2,2}, C \in N_{3,4}\}$$

$$\cup \{A \in N \mid \exists B, C \in N : A \rightarrow BC \in P, B \in N_{2,3}, C \in N_{4,4}\}$$

$\vdots$

Applying the CYK Algorithm

Example C.15

- $G : S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1						
2	<i>X</i>					
3	<i>X</i>	<i>X</i>				
4	<i>X</i>	<i>X</i>	<i>X</i>			
5	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>		
6	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	

Applying the CYK Algorithm

Example C.15

- $G : S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$					
2	X					
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X		
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$					
2	X	$\{B\}$				
3	X	X	$\{S\}$			
4	X	X	X	$\{S\}$		
5	X	X	X	X	$\{B\}$	
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

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- $G: S \rightarrow SA \mid a$
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	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$				
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$				
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G : S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
$i \backslash j$	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$			
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$			
2	X	$\{B\}$	$\{A, B\}$	$\{A\}$		
3	X	X	$\{S\}$	$\emptyset$		
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	{ <i>S</i> }	∅	{ <i>S</i> }			
2	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }	{ <i>A</i> , <i>B</i> }		
3	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅		
4	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	{ <i>S</i> }
5	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }
6	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \ j</i>	1	2	3	4	5	6
1	{ <i>S</i> }	∅	{ <i>S</i> }			
2	<i>X</i>	{ <i>B</i> }	{ <i>A, B</i> }	{ <i>A, B</i> }		
3	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	∅	
4	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	{ <i>S</i> }
5	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>B</i> }	{ <i>A, B</i> }
6	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }

Applying the CYK Algorithm

Example C.15

- $G : S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$		
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	{ <i>S</i> }	∅	{ <i>S</i> }	{ <i>S</i> }		
2	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }	{ <i>A</i> , <i>B</i> }	{ <i>B</i> }	
3	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	∅	
4	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	{ <i>S</i> }
5	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }
6	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	$\{S\}$	$\emptyset$	$\{S\}$	$\{S\}$		
2	X	$\{B\}$	$\{A, B\}$	$\{A, B\}$	$\{B\}$	
3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
4	X	X	X	$\{S\}$	$\emptyset$	$\{S\}$
5	X	X	X	X	$\{B\}$	$\{A, B\}$
6	X	X	X	X	X	$\{S\}$

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
1	{S}	∅	{S}	{S}	∅	
2	X	{B}	{A, B}	{A, B}	{B}	
3	X	X	{S}	∅	∅	∅
4	X	X	X	{S}	∅	{S}
5	X	X	X	X	{B}	{A, B}
6	X	X	X	X	X	{S}

Applying the CYK Algorithm

Example C.15

- $G: S \rightarrow SA \mid a$
 $A \rightarrow BS$
 $B \rightarrow BB \mid BS \mid b \mid c$
- $w = abaaba$

	<i>a</i>	<i>b</i>	<i>a</i>	<i>a</i>	<i>b</i>	<i>a</i>
<i>i \setminus j</i>	1	2	3	4	5	6
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2	X	{B}	{A, B}	{A, B}	{B}	{A}
3	X	X	{S}	∅	∅	∅
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1	{ <i>S</i> }	∅	{ <i>S</i> }	{ <i>S</i> }	∅	
2	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }	{ <i>A</i> , <i>B</i> }	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }
3	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	∅	∅
4	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	{ <i>S</i> }
5	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>B</i> }	{ <i>A</i> , <i>B</i> }
6	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	<i>X</i>	{ <i>S</i> }

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3	<i>X</i>	<i>X</i>	{ <i>S</i> }	∅	∅	∅
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3	X	X	$\{S\}$	$\emptyset$	$\emptyset$	$\emptyset$
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6	X	X	X	X	X	$\{S\}$

$$S \in N_{1,6} \implies w = abaaba \in L(G)$$

Summary: The Word Problem for Context-Free Languages

Seen:

- Given CFG G and $w \in \Sigma^*$, decide whether $w \in L(G)$ or not
- Decidable using CYK algorithm (based on dynamic programming)
- Cubic complexity

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Next:

- Emptiness problem

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- Important problem with many applications
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 - ...
- For regular languages this was easy: check in the corresponding DFA whether some final state is reachable from the initial state.
- Here: test whether start symbol is **productive**, i.e., whether it generates a terminal word

The Emptiness Test

Algorithm C.16 (Emptiness Test)

Input: $G = \langle N, \Sigma, P, S \rangle$

Question: $L(G) = \emptyset$?

Procedure: mark every $a \in \Sigma$ as productive;

 repeat

 if there is $A \rightarrow \alpha \in P$ such that all symbols in α productive then
 mark A as productive

 end

 until no further productive symbols found;

Output: “no” if S productive, otherwise “yes”

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Summary: The Emptiness Problem for Context-Free Languages

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- Emptiness problem decidable based on productivity of symbols

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Next:

- Closure properties of CFLs

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Theorem C.18

The set of CFLs is closed under concatenation, union, and iteration.

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For $i = 1, 2$, let $G_i = \langle N_i, \Sigma, P_i, S_i \rangle$ with $L_i := L(G_i)$ and $N_1 \cap N_2 = \emptyset$, and let $S \notin N_1 \cup N_2$ be a fresh nonterminal. Then

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- $L_1 \cdot L_2$ is generated by $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1 \cup N_2$ and

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- L_1^* is generated by $G := \langle N, \Sigma, P, S \rangle$ with $N := \{S\} \cup N_1$ and

$$P := \{S \rightarrow \varepsilon \mid S_1 S\} \cup P_1$$



Negative Results

Theorem C.19

The set of CFLs is not closed under intersection and complement.

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- Intersection: both

$$L_1 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\} \quad (\text{generated by } S \rightarrow AC, A \rightarrow aAb \mid \varepsilon, C \rightarrow Cc \mid \varepsilon)$$

and

$$L_2 := \{a^k b^l c^l \mid k, l \in \mathbb{N}\} \quad (\text{generated by } S \rightarrow AB, A \rightarrow aA \mid \varepsilon, B \rightarrow bBc \mid \varepsilon)$$

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(without proof).

- Complement: if CFLs were closed under complement, then also under intersection (as $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$).



Overview of Decidability and Closure Results

Decidability Results			
Class	$w \in L$	$L = \emptyset$	$L_1 = L_2$
Reg	+	+	+
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Closure Results					
Class	$L_1 \cdot L_2$	$L_1 \cup L_2$	$L_1 \cap L_2$	$\overline{L}$	L^*
Reg	+	+	+	+	+
CFL	+	+	—	—	+

Summary: Closure Properties of Context-Free Languages

Seen:

- Closure under concatenation, union and iteration
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Next:

- Automata model for CFLs

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Pushdown Automata I

- **Goal:** introduce an automata model which **exactly accepts CFLs**
- **Clear:** DFA not sufficient
(missing “counting capability”, e.g. for $\{a^n b^n \mid n \geq 1\}$)

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- **Clear:** DFA not sufficient
(missing “counting capability”, e.g. for $\{a^n b^n \mid n \geq 1\}$)
- DFA will be extended to **pushdown automata** by
 - adding a pushdown store which stores symbols from a pushdown alphabet and uses a special bottom symbol
 - adding push and pop operations to transitions

Pushdown Automata II

Definition C.20

A **pushdown automaton (PDA)** is of the form $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ where

- Q is a finite set of **states**
- Σ is the (finite) **input alphabet**
- Γ is the (finite) **pushdown alphabet**
- $\Delta \subseteq (Q \times \Gamma \times \Sigma_\epsilon) \times (Q \times \Gamma^*)$ is a finite set of **transitions**
- $q_0 \in Q$ is the **initial state**
- Z_0 is the **(pushdown) bottom symbol**
- $F \subseteq Q$ is a set of **final states**

Interpretation of $((q, Z, x), (q', \delta)) \in \Delta$: if the PDA $\mathfrak{A}$ is in state q where Z is on top of the stack and x is the next input symbol (or empty), then $\mathfrak{A}$ reads x , replaces Z by δ , and changes into the state q' .

Configurations, Runs, Acceptance

Definition C.21

Let $\mathfrak{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ be a PDA.

- An element of $Q \times \Gamma^* \times \Sigma^*$ is called a **configuration** of $\mathfrak{A}$.
- The **initial configuration** for input $w \in \Sigma^*$ is given by (q_0, Z_0, w) .
- The set of **final configurations** is given by $F \times \{\varepsilon\} \times \{\varepsilon\}$.
- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.

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- If $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$ for every $\gamma \in \Gamma^*$, $w \in \Sigma^*$.
- $\mathcal{A}$ **accepts** $w \in \Sigma^*$ if $(q_0, Z_0, w) \vdash^* (q, \varepsilon, \varepsilon)$ for some $q \in F$.
- The **language accepted by** $\mathcal{A}$ is $L(\mathcal{A}) := \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$.
- A language L is called **PDA-recognisable** if $L = L(\mathcal{A})$ for some PDA $\mathcal{A}$.
- Two PDA $\mathcal{A}_1, \mathcal{A}_2$ are called **equivalent** if $L(\mathcal{A}_1) = L(\mathcal{A}_2)$.

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

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 - $((q_0, Z, b), (q_1, \varepsilon))$ read first b
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- Observation: no transitions for
 - (q_0, Z_0, b) : input must start with a
 - (q_1, Z, a) : no a 's following b 's
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 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$(q_0, Z_0, aabb)$

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
 - q_0 : construction of PD while reading a 's
 - q_1 : deconstruction while reading b 's
 - q_2 : accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
 - Z_0 = bottom
 - $\#Z$ on PD = $\#a - \#b$ read so far
- $F = \{q_2\}$
- Δ :
 - $((q_0, Z_0, a), (q_0, ZZ_0))$ read first a
 - $((q_0, Z, a), (q_0, ZZ))$ read following a 's
 - $((q_0, Z, b), (q_1, \varepsilon))$ read first b
 - $((q_1, Z, b), (q_1, \varepsilon))$ read following b 's
 - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ change to final state
- Observation: no transitions for
 - (q_0, Z_0, b) : input must start with a
 - (q_1, Z, a) : no a 's following b 's
 - (q_1, Z_0, b) : more b 's than a 's
 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb)$$

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
 - q_0 : construction of PD while reading a 's
 - q_1 : deconstruction while reading b 's
 - q_2 : accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
 - Z_0 = bottom
 - $\#Z$ on PD = $\#a - \#b$ read so far
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 - $((q_0, Z_0, a), (q_0, ZZ_0))$ read first a
 - $((q_0, Z, a), (q_0, ZZ))$ read following a 's
 - $((q_0, Z, b), (q_1, \varepsilon))$ read first b
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 - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ change to final state
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 - (q_0, Z_0, b) : input must start with a
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 - (q_1, Z_0, b) : more b 's than a 's
 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb)$$

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
 - q_0 : construction of PD while reading a 's
 - q_1 : deconstruction while reading b 's
 - q_2 : accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
 - Z_0 = bottom
 - $\#Z$ on PD = $\#a - \#b$ read so far
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 - $((q_0, Z_0, a), (q_0, ZZ_0))$ read first a
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 - $((q_0, Z, b), (q_1, \varepsilon))$ read first b
 - $((q_1, Z, b), (q_1, \varepsilon))$ read following b 's
 - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ change to final state
- Observation: no transitions for
 - (q_0, Z_0, b) : input must start with a
 - (q_1, Z, a) : no a 's following b 's
 - (q_1, Z_0, b) : more b 's than a 's
 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb) \vdash (q_1, ZZ_0, b)$$

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
 - q_0 : construction of PD while reading a 's
 - q_1 : deconstruction while reading b 's
 - q_2 : accepting state
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, Z\}$
 - Z_0 = bottom
 - $\#Z$ on PD = $\#a - \#b$ read so far
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 - $((q_0, Z_0, a), (q_0, ZZ_0))$ read first a
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 - $((q_0, Z, b), (q_1, \varepsilon))$ read first b
 - $((q_1, Z, b), (q_1, \varepsilon))$ read following b 's
 - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ change to final state
- Observation: no transitions for
 - (q_0, Z_0, b) : input must start with a
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 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb) \vdash (q_1, ZZ_0, b) \vdash (q_1, Z_0, \varepsilon)$$

Examples of PDA I

Example C.22 (PDA for $L = \{a^n b^n \mid n \geq 1\}$)

$\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q = \{q_0, q_1, q_2\}$
 - q_0 : construction of PD while reading a 's
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 - $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ change to final state
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 - (q_1, Z_0, b) : more b 's than a 's
 - ...

Accepting run of PDA for input $w = aabb$:

(remember: if $((q, Z, x), (q', \delta)) \in \Delta$, then $(q, Z\gamma, xw) \vdash (q', \delta\gamma, w)$)

$$(q_0, Z_0, aabb) \vdash (q_0, ZZ_0, abb) \vdash (q_0, ZZZ_0, bb) \vdash (q_1, ZZ_0, b) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{Q}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba)$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

• $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)

$((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)

$((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)

$((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)

$((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba)$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba)$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
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- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
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- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
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- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$

Observation: $\mathcal{A}$ is **nondeterministic** – in a configuration of the form (q_0, cv, cw) ($c \in \Sigma, v, w \in \Sigma^*$), both (1) and (2) are applicable.

Examples of PDA II

Example C.23 (PDA for $L = \{ww^R \mid w \in \{a, b\}^*\}$ (palindromes of even length))

- Idea:
1. $\mathcal{A}$ pushes input w
 2. switches nondeterministically to the w^R recognition phase
 3. compares w and w^R symbol-wise by matching steps
 4. accepts with empty pushdown

Formally: $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{a, b\}$
- $\Gamma = \{Z_0, a, b\}$
- $F = \{q_2\}$

- $\Delta: ((q_0, Z, c), (q_0, cZ))$ for $Z \in \Gamma, c \in \Sigma$ (1)
- $((q_0, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (2)
- $((q_0, Z_0, \varepsilon), (q_1, Z_0))$ (2)
- $((q_1, c, c), (q_1, \varepsilon))$ for $c \in \Sigma$ (3)
- $((q_1, Z_0, \varepsilon), (q_2, \varepsilon))$ (4)

Accepting run of PDA for input $w = abba$:

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_1, aZ_0, a) \vdash (q_1, Z_0, \varepsilon) \vdash (q_2, \varepsilon, \varepsilon)$$

Observation: $\mathcal{A}$ is **nondeterministic** – in a configuration of the form (q_0, cv, cw) ($c \in \Sigma, v, w \in \Sigma^*$), both (1) and (2) are applicable. This yields **rejecting** runs, e.g.,

$$(q_0, Z_0, abba) \vdash (q_0, aZ_0, bba) \vdash (q_0, baZ_0, ba) \vdash (q_0, bbaZ_0, a) \vdash (q_0, abbaZ_0, \varepsilon) \not\vdash$$

Deterministic PDA

Definition C.24

A PDA $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called **deterministic (DPDA)** if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x) -transition in Δ and
2. if there is a (q, Z, a) -transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε) -transition in Δ .

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, xw) \vdash (q'_2, \delta_2 \gamma, w)$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, aw) \vdash (q'_2, \delta_2 \gamma, aw)$

Deterministic PDA

Definition C.24

A PDA $\mathcal{A} = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is called **deterministic (DPDA)** if for every $q \in Q, Z \in \Gamma$,

1. for every $x \in \Sigma_\varepsilon$, there is at most one (q, Z, x) -transition in Δ and
2. if there is a (q, Z, a) -transition in Δ for some $a \in \Sigma$, then there is no (q, Z, ε) -transition in Δ .

Remark: this excludes two types of nondeterminism:

1. if $((q, Z, x), (q'_1, \delta_1)), ((q, Z, x), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, xw) \vdash (q'_2, \delta_2 \gamma, w)$
2. if $((q, Z, a), (q'_1, \delta_1)), ((q, Z, \varepsilon), (q'_2, \delta_2)) \in \Delta$:
 $(q'_1, \delta_1 \gamma, w) \vdash (q, Z \gamma, aw) \vdash (q'_2, \delta_2 \gamma, aw)$

Corollary C.25

In a DPDA, every configuration has at most one $\vdash$ -successor.

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages
(DPDA-recognisable languages are **closed under complement**, which is generally not true for PDA-recognisable languages)

Expressiveness of DPDA

One can show: determinism restricts the set of acceptable languages (DPDA-recognisable languages are **closed under complement**, which is generally not true for PDA-recognisable languages)

Example C.26

The set of palindromes of even length is PDA-recognisable, but not DPDA-recognisable (without proof).

Summary: Pushdown Automata

Seen:

- Extension of finite automata by pushdown store
- Enables “counting” (e.g., $\{a^n b^n \mid n \geq 1\}$)
- Determinism restricts expressivity (in contrast to finite automata)

Summary: Pushdown Automata

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- Extension of finite automata by pushdown store
- Enables “counting” (e.g., $\{a^n b^n \mid n \geq 1\}$)
- Determinism restricts expressivity (in contrast to finite automata)

Next:

- Relation between PDA and context-free languages

Outline of Part C

Context-Free Grammars and Languages

Context-Free vs. Regular Languages

Chomsky Normal Form

The Word Problem for Context-Free Languages

The Emptiness Problem for Context-Free Languages

Closure Properties of Context-Free Languages

Pushdown Automata

Pushdown Automata and Context-Free Languages

Theorem C.27

A language is context-free iff it is PDA-recognisable.

PDA and Context-Free Languages I

Theorem C.27

A language is context-free iff it is PDA-recognisable.

Proof.

“ $\Leftarrow$ ”: omitted

“ $\Rightarrow$ ”: let $G = \langle N, \Sigma, P, S \rangle$ be a CFG. Construction of PDA $\mathcal{A}_G$ recognising $L(G)$:

- $\mathcal{A}_G$ simulates a derivation of G where always the leftmost nonterminal of a sentence is replaced (“leftmost derivation”)
- begin with S on pushdown
- if nonterminal on top: apply a corresponding production rule
- if terminal on top: match with next input symbol

(cf. formal construction on following slide)



PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
- $Z_0 := S$
- for each $A \rightarrow \alpha \in P$: $((q_0, A, \varepsilon), (q_0, \alpha)) \in \Delta$ (“expansion”)
- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ (“matching”)
- $F := Q$



PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

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□

Example C.28 (“Bracket language” given by $G : S \rightarrow \langle \rangle \mid \langle S \rangle \mid SS$)

$\mathfrak{A}_G = \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ with

- $Q = F = \{q_0\}$
- $\Sigma = \{\langle, \rangle\}, \Gamma = \{S, \langle, \rangle\}$
- $Z_0 = S$
- Δ :
 $((q_0, S, \varepsilon), (q_0, \langle \rangle))$ $((q_0, \langle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, \langle S \rangle))$ $((q_0, \rangle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, SS))$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

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 $((q_0, S, \varepsilon), (q_0, \langle S \rangle)) \quad ((q_0, \rangle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, SS))$

Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:
 $(q_0, S, \langle \rangle \langle \rangle \langle \rangle)$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

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 $((q_0, S, \varepsilon), (q_0, \langle S \rangle)) \quad ((q_0, \rangle, \rangle), (q_0, \varepsilon))$
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Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

$$(q_0, S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, SS, \langle \rangle \langle \rangle \langle \rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

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 $((q_0, \textcolor{red}{S}, \varepsilon), (q_0, \langle \textcolor{red}{S} \rangle)) \quad ((q_0, \rangle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, SS))$

Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

$$(q_0, S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \textcolor{red}{S}S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \langle \textcolor{red}{S} \rangle S, \langle \rangle \langle \rangle \langle \rangle)$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

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Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

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PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

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Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

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PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

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PDA and Context-Free Languages II

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PDA and Context-Free Languages II

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Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

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 \vdash (q_0, \langle \rangle \rangle S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \rangle \rangle S, \rangle \rangle \langle \rangle) \vdash (q_0, \rangle S, \rangle \langle \rangle) \vdash (q_0, S, \langle \rangle)
 \end{array}$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

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Example C.28 (“Bracket language” given by $G : S \rightarrow \langle \rangle \mid \langle S \rangle \mid SS$)

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- $Z_0 = S$
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 $((q_0, S, \varepsilon), (q_0, \langle S \rangle))$ $((q_0, \rangle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, SS))$

Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

$$\begin{aligned}
 & (q_0, S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, SS, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \langle S \rangle S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, S \rangle S, \langle \rangle \langle \rangle \langle \rangle) \\
 & \vdash (q_0, \langle \rangle \rangle S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \rangle \rangle S, \rangle \rangle \langle \rangle \vdash (q_0, \rangle S, \rangle \langle \rangle) \vdash (q_0, S, \langle \rangle) \\
 & \vdash (q_0, \langle \rangle, \langle \rangle)
 \end{aligned}$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

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Example C.28 (“Bracket language” given by $G : S \rightarrow \langle \rangle \mid \langle S \rangle \mid SS$)

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 $((q_0, S, \varepsilon), (q_0, \langle S \rangle)) \quad ((q_0, \rangle, \rangle), (q_0, \varepsilon))$
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Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

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 (q_0, S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, SS, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \langle S \rangle S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, S \rangle S, \langle \rangle \langle \rangle \langle \rangle) \\
 \vdash (q_0, \langle \rangle \rangle S, \langle \rangle \langle \rangle \langle \rangle) \vdash (q_0, \rangle \rangle S, \rangle \rangle \langle \rangle) \vdash (q_0, \rangle S, \rangle \langle \rangle) \vdash (q_0, S, \langle \rangle) \\
 \vdash (q_0, \langle, \rangle), \langle \rangle) \vdash (q_0, \rangle, \rangle)
 \end{array}$$

PDA and Context-Free Languages II

Proof of Theorem C.27 (continued).

“ $\Rightarrow$ ”: Formally, $\mathfrak{A}_G := \langle Q, \Sigma, \Gamma, \Delta, q_0, Z_0, F \rangle$ is given by

- $Q := \{q_0\}$
- $\Gamma := N \cup \Sigma$
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- for each $a \in \Sigma$: $((q_0, a, a), (q_0, \varepsilon)) \in \Delta$ (“matching”)
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□

Example C.28 (“Bracket language” given by $G : S \rightarrow \langle \rangle \mid \langle S \rangle \mid SS$)

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- Δ : $((q_0, S, \varepsilon), (q_0, \langle \rangle)) \quad ((q_0, \langle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, \langle S \rangle)) \quad ((q_0, \rangle, \rangle), (q_0, \varepsilon))$
 $((q_0, S, \varepsilon), (q_0, SS))$

Accepting run for input $w = \langle \rangle \langle \rangle \langle \rangle$:

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 (q_0, S, \langle \rangle \langle \rangle \langle \rangle) & \vdash & (q_0, SS, \langle \rangle \langle \rangle \langle \rangle) & \vdash & (q_0, \langle S \rangle S, \langle \rangle \langle \rangle \langle \rangle) & \vdash & (q_0, S \rangle S, \langle \rangle \langle \rangle \langle \rangle) \\
 \vdash & (q_0, \langle \rangle \rangle S, \langle \rangle \langle \rangle \langle \rangle) & \vdash & (q_0, \rangle \rangle S, \rangle \rangle \langle \rangle) & \vdash & (q_0, \rangle S, \rangle \langle \rangle) & \vdash & (q_0, S, \langle \rangle) \\
 \vdash & (q_0, \langle \rangle, \langle \rangle) & \vdash & (q_0, \rangle, \rangle) & \vdash & (q_0, \varepsilon, \varepsilon)
 \end{array}$$

Summary: Pushdown Automata and Context-Free Languages

Seen:

- Construction of PDA for given CFG ($\Rightarrow$ parser generation!)
- Reverse direction also possible
- Thus: PDA and CFG equivalent

Summary: Pushdown Automata and Context-Free Languages

Seen:

- Construction of PDA for given CFG ($\Rightarrow$ parser generation!)
- Reverse direction also possible
- Thus: PDA and CFG equivalent

Outlook:

- **Equivalence problem** for CFG and PDA (“ $L(X_1) = L(X_2)$?”): generally undecidable, but decidable for DPDA
- **Pumping Lemma** for CFL (e.g., to prove that $\{a^n b^n c^n \mid n \geq 1\}$ not context-free)
- **Greibach Normal Form** for CFG
- Systematic construction of **deterministic and efficient** parsers for compilers (LL/LR grammars)
- Non-context-free grammars and languages (e.g., **context-sensitive** languages such as $\{a^n b^n c^n \mid n \geq 1\}$)