

# Exercises (Context-Free Languages)

## C1: Construction of Context-Free Grammars

**Task:** Give context-free grammars that generate the following languages.

(a)  $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b)  $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c)  $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d)  $L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd, } a \text{ in middle position}\} (= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

(e)  $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

G:  $S \rightarrow a S b \mid \epsilon$

$$S \Rightarrow a S b \Rightarrow a a S b b$$

$$\Rightarrow a^n S b^n \Rightarrow a^n b^n$$

$$\Rightarrow L(G) = \{a^n b^n \mid n \geq 0\} \quad a^k b^k$$

(a)  $L_1 = \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

$$S \xrightarrow{(1)} a A c \mid B$$

$$A \xrightarrow{(2,3)} a A c \mid B$$

$$B \xrightarrow{(4,5)} b B c \mid \epsilon$$

$$S \xrightarrow{(1)} a A c$$

$$(2) \quad k \rightarrow a^k A c^k$$

$$(3) \quad a^k B c^k$$

$$(4) \quad a^k b^l B c^l c^k$$

$$(5) \quad a^k b^l c^{k+l}$$

Simplified:

$$S \rightarrow a S c \mid B$$

$$B \rightarrow b B c \mid \epsilon$$

$$(b) L_2 = \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$$

$$= L_2' \cdot L_2''$$

$$L_2' = \{a^k b^k \mid k \in \mathbb{N}\}$$

$$L_2'' = \{c^l d^l \mid l \in \mathbb{N}\}$$

$$S \rightarrow S_1 S_2$$

$$S_1 \rightarrow a S_1 b \mid \epsilon \quad S_2 \rightarrow c S_2 d \mid \epsilon$$

$$(d) L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd, } a \text{ in middle position}\}$$

$$= \{uav \mid u, v \in \{a, b\}^*, |u| = |v|\}$$

$$S \rightarrow a \mid a S a \mid a S b \mid b S a \mid b S b$$

Alternative:

$$S \rightarrow a \mid S_1 S S_1$$

$$S_1 \rightarrow a \mid b$$

$$S \Rightarrow S_1 S S_1$$

$$\Rightarrow a S S_1$$

$$\Rightarrow a S b \Rightarrow aab \quad \checkmark$$

## C2: From Regular to Context-Free Languages

**Task:** Show that every regular expression can directly be translated into an equivalent context-free grammar.

Seen in course:  $DFA \leadsto CFG$

$$q \xrightarrow{a} q' \leadsto q \rightarrow a q'$$

$$(q_f) \leadsto q_f \rightarrow \epsilon$$

$$- \alpha = \emptyset: L(\alpha) = \emptyset: P = \emptyset \text{ or } P = \{S \rightarrow \emptyset\}$$

$$- \alpha = \Sigma: L(\alpha) = \{\epsilon\}: P = \{S \rightarrow \epsilon\}$$

$$- \alpha = a: L(\alpha) = \{a\}, P = \{S \rightarrow a\}$$

$$- \alpha = \alpha_1 / \alpha_2: L(\alpha) = L(\alpha_1) \cup L(\alpha_2)$$

$$P = \{S \rightarrow S_1 / S_2\} \cup P_1 \cup P_2 \quad \leftarrow \text{for } \alpha_1 / \alpha_2$$

$$- \alpha = \alpha_1 \cdot \alpha_2: L(\alpha) = L(\alpha_1) \cdot L(\alpha_2)$$

$$P = \{S \rightarrow S_1 S_2\} \cup P_1 \cup P_2$$

$$- \alpha = \alpha_1^*: L(\alpha) = (L(\alpha_1))^*$$

$$P = \{S \rightarrow \epsilon / S S_1\} \cup P_1$$

$$\text{For CFG } G: L(G) = \{w \in \Sigma^* / S \Rightarrow^* w\}$$

Example for construction (C2):

$$\alpha = (aa|b)^*, \quad aab \in L(\alpha)$$

$$\alpha = \alpha_n^*, \quad \alpha_n = aa|b$$

$$\Rightarrow \Sigma_{(1,2)} \Sigma_1 \Sigma_n$$

$$\alpha_n = \alpha_2 | \alpha_3, \quad \alpha_2 = aa, \quad \alpha_3 = b$$

$$\Rightarrow \Sigma_{(2,4)} \Sigma_2 | \Sigma_3$$

$$\alpha_2 = \alpha_4 \cdot \alpha_5, \quad \alpha_4 = a, \quad \alpha_5 = a$$

$$\Sigma_2 \stackrel{(5)}{=} \Sigma_4 \cdot \Sigma_5$$

$$\alpha_3 = b$$

$$\Rightarrow \Sigma_3 \stackrel{(6)}{\rightarrow} b$$

$$\alpha_4 = a = \alpha_5$$

$$\Sigma_4 \stackrel{(7)}{\rightarrow} a, \quad \Sigma_5 \stackrel{(8)}{\rightarrow} a$$

$$\begin{array}{c} \Sigma \stackrel{(2)}{=} \Sigma \Sigma \\ \stackrel{(2)}{=} \Sigma \Sigma \Sigma \\ \stackrel{(4)}{=} \Sigma_1 \Sigma_1 \\ \stackrel{(2)}{=} \Sigma_2 \Sigma_2 \\ \stackrel{(5)}{=} \Sigma_4 \Sigma_5 \\ \stackrel{(7,8)}{=} \Sigma_4 \Sigma_5 \Sigma_5 \\ \quad \quad \quad a a \Sigma_n \end{array}$$

(4)  $\Rightarrow aa\beta$

(6)  $\Rightarrow aab \quad \checkmark$

## C4: The Word Problem for Context-Free Languages

**Task:** Let  $G$  be the following context-free grammar:

$$\begin{aligned} S &\rightarrow AB \mid BC \\ A &\rightarrow BA \mid a \\ B &\rightarrow CC \mid b \\ C &\rightarrow AB \mid a \end{aligned}$$

and let  $w := baaba$ . Employ the CYK-Algorithm to show that  $w \in L(G)$ . Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \quad (1 \leq i \leq j \leq 5)$$

where  $w[i,j] := a_i \dots a_j$  for  $w = a_1 a_2 a_3 a_4 a_5$ .

| $i \backslash j$ | 1 | 2     | 3            | 4            | 5       |
|------------------|---|-------|--------------|--------------|---------|
| 1                | B | A, S  | <del>Ø</del> | <del>Ø</del> | S, A, C |
| 2                | X | A', C | B            | B            | S, C, A |
| 3                | X | X     | A, C         | S, C         | B       |
| 4                | X | X     | X            | B            | S, A    |
| 5                | X | X     | X            | X            | A', C   |

$$\begin{aligned} w[1,3] &= w[1,1] \cdot w[2,3] = b \cdot aa \\ &= w[1,2] \cdot w[3,3] = ba \cdot a \end{aligned}$$

$$S \in N_{1,5} = \{A \in N \mid S \Rightarrow^* baaba\}$$

$$\Rightarrow baaba \in L(G) \quad \checkmark$$

Corresponding derivation:

$$S \Rightarrow AB \Rightarrow BAB \Rightarrow bAB$$

$$\Rightarrow baB \Rightarrow baCC \Rightarrow baABC$$

$$\Rightarrow baABC \Rightarrow baaba$$