

Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd, } a \text{ in middle position}\} (= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$

$$G = (N, \Sigma, P, S)$$

$$G: S \rightarrow a S b \mid \epsilon$$

$$S \Rightarrow a S b \Rightarrow a a S b b \Rightarrow \dots \Rightarrow a^n S b^n \\ \Rightarrow a^n b^n$$

$$\leadsto L(G) = \{a^n b^n \mid n \geq 0\}$$

(a) $S \xrightarrow{(1,2)} a S c \mid b S c \mid A$

$S \Rightarrow b S c \Rightarrow b a S c c \hookrightarrow$

$A \xrightarrow{(3,4)} b A c \mid \epsilon$

$S \xrightarrow{k \times (1)} a^k S c^k \xrightarrow{(2)} a^k A c^k$

$\xrightarrow{l \times (3)} a^k b^l A c^l c^k \xrightarrow{(4)} a^k b^l c^{k+l}$

(b) $S \rightarrow AB$

$A \rightarrow aAb \mid \epsilon$

$B \rightarrow cBd \mid \epsilon$

$L_1 \cdot L_2 = \{vw \mid v \in L_1, w \in L_2\}$

$$(c) S \rightarrow AB$$

$$A \rightarrow aAb \mid ab$$

$$B \rightarrow bB \mid b$$

To generate $a^k b^l$ ($k \geq 1, l \geq k$):

$$S \Rightarrow AB \xRightarrow{k-1} a^{k-1} A b^{k-1} B \Rightarrow a^k b^k B$$

$$\xRightarrow{l-k} a^k b^k b^{l-k-1} B$$

$$\Rightarrow a^k b^k b^{l-k} = a^k b^l$$

$$(d) S \rightarrow aSa \mid aSb \mid bSa \mid bSb \mid a$$

Alternative: $S \rightarrow A S A \mid a$

$$A \rightarrow a \mid b$$

$$(e) S \rightarrow A \mid B \quad (\text{alternative})$$

$$A \rightarrow CD$$

$$C \rightarrow \underline{a} C \underline{b} \mid \epsilon \quad \left. \vphantom{C \rightarrow \underline{a} C \underline{b} \mid \epsilon} \right\} k=l$$

$$D \rightarrow cD \mid \epsilon$$

$$B \rightarrow \underline{a} B \underline{c} \mid \epsilon \quad \left. \vphantom{B \rightarrow \underline{a} B \underline{c} \mid \epsilon} \right\} k=m$$

$$E \rightarrow \underline{b} E \mid \epsilon$$

Not context-free:

$$\{a^n b^n c^n \mid n \geq 0\}$$

C2: From Regular to Context-Free Languages

Task: Show that every regular expression can directly be translated into an equivalent context-free grammar.

In course: DFA $\alpha \rightsquigarrow$ CFG G_α
such $L(\alpha) = L(G_\alpha)$
 $q \xrightarrow{a} q'$ in $\alpha \rightsquigarrow q \rightarrow aq'$ in G_α
 (q_f) in $\alpha \rightsquigarrow q_f \rightarrow \epsilon$

Solution: Given reg. expr. α over Σ
we inductively construct CFG
 $G_\alpha = (N, \Sigma, P, \delta)$ with $L(G_\alpha) = L(\alpha)$.

- $\alpha = \emptyset$: $P = \{\delta \rightarrow \delta\}$ $L = \emptyset$

- $\alpha = \Sigma$: $P = \{\delta \rightarrow \Sigma\}$ $L = \{\Sigma\}$

- $\alpha = a \in \Sigma$: $P = \{\delta \rightarrow a\}$

- $\alpha = \alpha_1 \mid \alpha_2$: $P = \{\delta \rightarrow \delta_1 \mid \delta_2\} \cup$
 $P_1 \cup P_2$

where $G_{\alpha_i} = (N_i, \Sigma, P_i, \delta_i)$ for $i=1,2$

- $\alpha = \alpha_1 \cdot \alpha_2$: $P = \{\delta \rightarrow \delta_1 \delta_2\} \cup$
 $P_1 \cup P_2$

C3: Chomsky Normal Form

Task: Transform the following grammar into Chomsky Normal Form:

$$S \rightarrow xAx \mid CyBA \mid BE \mid z \mid xxx$$

$$A \rightarrow C \mid x$$

$$B \rightarrow A$$

$$C \rightarrow yyy \mid B$$

$$- \alpha = \alpha_n^* : P = \{ S \rightarrow S_n S \mid \epsilon \} \cup P_n$$

Example: $\alpha = (aa/b)^*$

$$\alpha = \alpha_n^* \text{ for } \alpha_n = aa/b$$

$$\Rightarrow S \rightarrow S_n S \mid \epsilon$$

Derivation of aab :
 $S \Rightarrow S_n S \Rightarrow S_2 S$

$$\alpha_n = \alpha_2 \mid \alpha_3$$

for $\alpha_2 = aa$ and

$$\Rightarrow S_1 \rightarrow S_2 \mid S_3 \mid \alpha_3 = b$$

$$\alpha_2 = \alpha_4 \cdot \alpha_5$$

for $\alpha_4 = \alpha_5 = a$

$$\Rightarrow S_2 \rightarrow S_4 S_5$$

$$\alpha_3 = b \Rightarrow S_3 \rightarrow b$$

$$\alpha_4 = \alpha_5 = a \Rightarrow S_4 \rightarrow a, S_5 \rightarrow a$$

$$\begin{aligned} &\Rightarrow S_4 S_5 S \\ &\Rightarrow a S_5 S \\ &\Rightarrow aa S \\ &\Rightarrow aa S_2 S \\ &\Rightarrow aab S \\ &\Rightarrow aab \end{aligned}$$

C4: The Word Problem for Context-Free Languages

Task: Let G be the following context-free grammar:

$$\begin{aligned} S &\rightarrow AB \mid BC \\ A &\rightarrow BA \mid a \\ B &\rightarrow CC \mid b \\ C &\rightarrow AB \mid a \end{aligned}$$

and let $w := baaba$. Employ the CYK-Algorithm to show that $w \in L(G)$. Use the following table to compute the sets

$$N_{i,j} := \{A \in N \mid A \Rightarrow^* w[i,j]\} \quad (1 \leq i \leq j \leq 5)$$

where $w[i,j] := a_i \dots a_j$ for $w = a_1 a_2 a_3 a_4 a_5$.

$i \backslash j$	1	2	3	4	5
1	B	A, S	$\emptyset$	$\emptyset$	S, A, C
2	X	A', C	B	B	S', A', C
3	X	X	A, C	S, C	B'
4	X	X	X	B'	A, S
5	X	X	X	X	A', C

$$w[1,2] = w[1,1] \cdot w[2,2]$$

$$w[1,3] = w[1,1] \cdot w[2,3]$$

$$= w[1,2] \cdot w[3,3]$$

$$\Rightarrow S \in N_{1,5} = \{A \in N \mid A \Rightarrow^* baaba\}$$

$$\Rightarrow baaba \in L(G) \checkmark$$

$$S \Rightarrow BC \Rightarrow bC \Rightarrow bAB \Rightarrow baB$$

$$\Leftrightarrow baCC \Rightarrow baABC \Rightarrow baabc$$

$$\Rightarrow baabC \Rightarrow baaba \checkmark$$