

Exercises (Context-Free Languages)

C1: Construction of Context-Free Grammars

Task: Give context-free grammars that generate the following languages.

(a) $L_1 := \{a^k b^l c^{k+l} \mid k, l \in \mathbb{N}\}$

(b) $L_2 := \{a^k b^k c^l d^l \mid k, l \in \mathbb{N}\}$

(c) $L_3 := \{a^k b^l \mid k \geq 1, l > k\}$

(d) $L_4 := \{w \in \{a, b\}^* \mid |w| \text{ odd, } a \text{ in middle position}\} (= \{uav \in \{a, b\}^* \mid |u| = |v|\})$

(e) $L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N}, k = l \text{ or } k = m\}$ ^{"minimal" word: abb}

(c) $L_3 = \{a^k b^l \mid k \geq 1, l > k\}$

$S \xrightarrow{(1,2)} a S b \mid a b B$

$B \xrightarrow{(3,4)} b B \mid \varepsilon b$

$S \Rightarrow a B \Rightarrow a$

$S \Rightarrow a b B \Rightarrow a b b \quad \checkmark$

$S \xrightarrow{(1)} k-1 \quad a^{k-1} S b^{k-1}$

$\xrightarrow{(2)} a^{k-1} a b B b^{k-1}$

$= a^k b B b^{k-1}$

$\xrightarrow{(3)} l-k-1 \quad a^k b b^{l-k-1} B b^{k-1}$

$$\begin{aligned} \underline{(4)} \Rightarrow a^k b^{l-k} b^k &= a^k b^l \quad \checkmark \end{aligned}$$

Remark: $L_3 = L_3' \cdot L_3''$

$$L_3' = \{a^n b^n \mid n \geq 1\} \quad L_3'' = b^+$$

$$(e) \quad L_5 = \{a^k b^l c^m \mid k, l, m \in \mathbb{N},$$

$$L_5 = L_5' \cup L_5'' \mid k=l \text{ or } k=m\}$$

$$S \xrightarrow{(1,2)} AC$$

$$A \xrightarrow{(2,4)} aAb \mid \Sigma$$

$$C \xrightarrow{(5,6)} cC \mid \Sigma$$

$$B \xrightarrow{(7,8)} aBc \mid D$$

$$D \xrightarrow{(9,10)} bD \mid \Sigma$$

$$a^k b^k c^m$$

$$\begin{aligned} L_5' : S &\xrightarrow{(1)} AC \xrightarrow{(3)} a^k A b^k C \xrightarrow{(4)} a^k b^k C \xrightarrow{(5)} a^k b^k c^m C \xrightarrow{(6)} a^k b^k c^m \end{aligned}$$

C2: From Regular to Context-Free Languages

Task: Show that every regular expression can directly be translated into an equivalent context-free grammar.

$$\begin{aligned} L_5^{(1)} : & \int \xrightarrow{(2)} B \xrightarrow{(7)} a^k B c^k \\ & \xrightarrow{(8)} a^k D c^k \xrightarrow{(9)} a^k b^l D c^k \\ & \xrightarrow{(10)} a^k b^l c^k \quad \checkmark \end{aligned}$$

C5: The Emptiness Problem for Context-Free Languages

Task: Using the marking algorithm, check whether the following grammars generate empty languages or not:

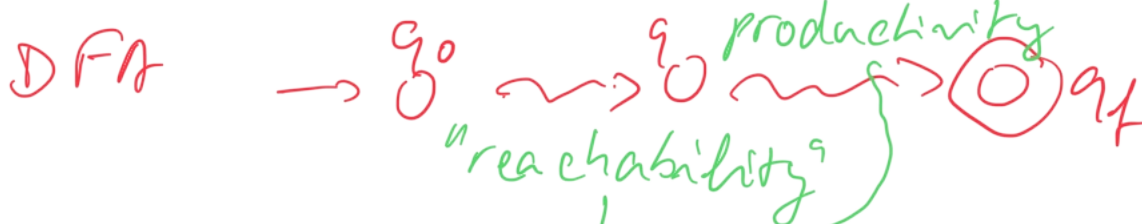
- (a) $S \rightarrow \underline{X}U \mid U\underline{W}$
 $U \rightarrow S\underline{a}W \mid V\underline{b}U \mid S\underline{b}X$
 $V \rightarrow U \mid \underline{a}\underline{b} \mid \underline{X}W$
 $W \rightarrow \underline{b}X\underline{a}U\underline{b} \mid V\underline{b}X \mid S\underline{X}$
 $X \rightarrow \underline{b}V \mid W\underline{X}$ $\Rightarrow L(G) = \emptyset$
 0. initialisation
 1st phase
 2nd phase
 3rd phase
 not productive

- (b) $S \rightarrow \underline{A}\underline{C} \mid \underline{D}\underline{A}$
 $A \rightarrow \underline{S}\underline{b}C \mid \underline{B}\underline{a}B \mid \underline{A}\underline{a}D$
 $B \rightarrow \underline{A} \mid \underline{b}\underline{a} \mid \underline{D}C$
 $C \rightarrow \underline{a}D\underline{b}A\underline{a} \mid \underline{B}\underline{a}D \mid \underline{S}D$
 $D \rightarrow \underline{a}B \mid \underline{C}D$
 0. initialisation
 1st phase
 2nd phase
 3rd phase

$\exists$ marked $\Rightarrow \exists$ productive
 $\Rightarrow L(G) \neq \emptyset$

Example derivation:

$S \Rightarrow DA \Rightarrow aBA \Rightarrow abaA$
 $\Rightarrow abcBaB \Rightarrow^2 abcbaaba$



CFG $S \Rightarrow^* \alpha X \beta \Rightarrow^* w \in \Sigma^*$
 $(\alpha, \beta \in (\Sigma \cup N)^*, X \in N, \Sigma)$

C8: From Context-Free Grammars to Pushdown Automata

Task: Construct a PDA that accepts the language generated by the grammar

$$S \rightarrow aSa \mid bSb \mid \varepsilon$$

(which is $L = \{ww^R \mid w \in \{a,b\}^*\}$, i.e., palindromes of even length).

(where $(a_n \dots a_1)^R = a_1 \dots a_n$)

In general: $\mathcal{M}_C = (Q, \Sigma, \Gamma, \Delta, q_0, z_0, F)$

$$- Q = \{q_0\} \quad - z_0 = \$$$

$$- \Gamma = N \cup \Sigma \quad - F = \{q_0\}$$

- for each $A \rightarrow \alpha \in P$:

$$((q_0, A, \Sigma), (q_0, \alpha)) \in \Delta \text{ ("expansion")}$$

- for each $a \in \Sigma$,

$$((q_0, a, a), (q_0, \Sigma)) \text{ ("matching")}$$

Initial config: (q_0, z_0, w)

Here:

$$- Q = \{q_0\} \quad - z_0 = \$$$

$$- \Gamma = N \cup \Sigma = \{\$, a, b\} \quad - F = \{q_0\}$$

$$- \Delta: ((q_0, \$, \Sigma), (q_0, a\$a))$$

$$((q_0, \$, \Sigma), (q_0, b\$b))$$

$$((q_0, \$, \Sigma), (q_0, \Sigma))$$

$((q_0, a, a), (q_0, \Sigma))$

$((q_0, b, b), (q_0, \Sigma))$

For example:

$(q_0, \Sigma, abbbba)$

$\vdash (q_0, a \Sigma a, abbbba)$

$\vdash (q_0, \Sigma a, bbbba)$

$\vdash (q_0, b \Sigma b a, bbbba)$

$\vdash (q_0, \Sigma b a, bbbba)$

$\vdash (q_0, b \Sigma b b a, bbbba)$

$\vdash (q_0, \Sigma b b a, bbbba)$

$\vdash (q_0, b b a, b b a)$

$\vdash^3 (q_0, \Sigma, \Sigma)$

✓

$(q_0, \Sigma, abbbba)$

$\vdash (q_0, a \Sigma a, abbbba)$

$\vdash (q_0, \Sigma a, bbbba)$

$\vdash (q_0, a \Sigma a a, bbbba)$

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